Topic Review
Precalculus Handout 1.2
Inequalities and Absolute Value

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Real numbers are ordered where given the real numbers a, b, and c:

$a < b$
a "**is less than**" b
Ex: $1 < 2$

$c > b$
c "**is greater than**" b
Ex: $3 > 2$

If a real number is ordered given that b is between a and c from the above example:

$a < b < c$
b "**is between**" a and c
Ex: $1 < 2 < 3$

In the cases above, the numbers are not included in the interval.

Inequality Properties

For real numbers a, b, c, and constant *k*.

1. Transitive Property (our above example)
If $a < b$ and $b < c$, then $a < c$
Similarly, if $a \le b$ and $b \le c$, then $a \le c$

2. Converse Property
If $a < b$, then $b > a$ (similarly, if $a \le b$, then $b \ge a$)
If $a > b$, then $b < a$ (similarly, if $a \ge b$, then $b \le a$)

3. Addition or Subtraction with a Constant

If $a < b$, then $a + k < b + k$ (similarly, if $a \le b$, then $a + k \le b + k$)
If $c > b$, then $c - k > b - k$ (similarly, if $c \ge b$, then $c - k \ge b - k$)

4. Multiplication or Division by a Positive Constant

If $a < b$, then $ak < bk$ (similarly, if $a \le b$, then $ak \le bk$)
If $c > b$, then $\frac{c}{k} > \frac{b}{k}$ (similarly, if $c \ge b$, then $\frac{c}{k} \ge \frac{b}{k}$)

5. Multiplication or Division by a Negative Constant

When you multiply or divide an inequality by a negative number, the inequality is <u>reversed</u>.

***Example 1.2a*: Given the inequality, $-3x \le 6$ solve for x.**

First we divide both sides by -3:
$$\frac{-3x}{-3} \le \frac{6}{-3}$$

Then we reverse the inequality:
$x \ge -2$

***Example 1.2b*: Given the inequality, $\frac{x}{-4} > 5$, solve for x.**

First we multiply both sides by -4:
$$\frac{x}{-4}(-4) > 5(-4)$$

Then we reverse the inequality:
$x < -20$

<u>Intervals on the real number line</u>
***For the graphs, see the accompanying inequality graph worksheet.**

In **open intervals**, the endpoints a and b are not included. They are denoted by parentheses (a, b) and when graphed on a number line, the circles are open/hollow.

In **closed intervals**, the endpoints a and b are included. They are denoted by square brackets [a, b] and when graphed on a number line, the circles are closed/solid.

All real numbers

Notation

Set: $(-\infty, \infty)$

Interval: $\{x \mid x \text{ is a real number}\}$

Bounded open interval

Notation

Set: (a, b)

Interval: $\{x \mid a < x < b\}$

Bounded closed interval

Notation

Set: $[a, b]$

Interval: $\{x \mid a \le x \le b\}$

Half-closed (half-open) intervals

Notation

Set: $[a, b)$
$(a, b]$

Interval: $\{x \mid a \le x < b\}$
$\{x \mid a < x \le b\}$

Unbounded open interval

Notation

Set: $(-\infty, b)$
(a, ∞)

Interval: $\{x \mid x < b\}$
$\{x \mid x > a\}$

Unbounded closed interval

Notation

Set: $(-\infty, b]$
$[a, \infty)$

Interval: $\{x \mid x \le b\}$
$\{x \mid x \ge a\}$

<u>Solving Inequalities</u>

***Graphs to the solutions for the examples are in the inequality graph worksheet.**

Example 1.2c

Solve the linear inequality $4x + 3 > 15$ and graph its solution on the real number line.

$4x + 3 > 15$

$4x + 3 - 3 > 15 - 3$ Subtract 3 from both sides.

$\frac{4x}{4} > \frac{12}{4}$ Divide both sides by 4.

$x > 3$ Solution

The solution is the set of all x's such that x is greater than 3; $\{x \mid x > 3\}$. In set notation: $(3, \infty)$. Remember, 3 is not included so on the graph it is denoted by an open circle. Since the values for x are greater than 3, it is unbounded.

Example 1.2d
Solve the linear inequality $-4 \le 2x + 6 \le 14$ and graph its solution on the real number line.

$-4 \le 2x + 6 \le 14$

$-4 - 6 \le 2x + 6 - 6 \le 14 - 6$ Subtract 6 from all parts.
$-10 \le 2x \le 8$

$-\frac{10}{2} \le \frac{2x}{2} \le \frac{8}{2}$ Divide all parts by 2.

$-5 \le x \le 4$ Solution

The solution set is $[-5, 4]$.

Example 1.2e
Solve the quadratic inequality $x^2 + 6x + 14 \le 6$ and graph its solution on the real number line.

$x^2 + 6x + 14 \le 6$

$x^2 + 6x + 14 - 6 \le 6 - 6$ Subtract 6 from both sides to change to general form.
$x^2 + 6x + 8 \le 0$ General form

$(x+ 4)(x+2) \le 0$ Factor

When we solve for x, we get $x = -4$ and $x = -2$. We have to test the signs in each test interval to solve the inequality.

There are three intervals that we need to test:

(-∞, -4], [-4, -2], and [-2, ∞)

Step 1: Choose a number in the interval and compute by replacing the x in the polynomial.

The polynomial: $x^2 + 6x + 8$

Interval	Number Chosen in Interval	Answer
(-∞, -4]	**-5**	3
[-4, -2]	**-3**	-1
[-2, ∞)	**0**	8

So for the interval (-∞, -4], we choose -5 and put it in the polynomial to test the sign.

$(-5)^2 + 6(-5) + 8$
25 - 30 + 8
3
The polynomial is positive for all real numbers in the interval (-∞, -4].

For the interval [-4, -2], we choose -3.
$(-3)^2 + 6(-3) + 8$
9 - 18 + 8
-1
The polynomial is negative for all real numbers in the interval [-4, -2].

For the interval [-2, ∞), we choose 0.
$(0)^2 + 6(0) + 8$
8
The polynomial is positive for all real numbers in the interval [-2, ∞).

Step 2: Now we look at the original inequality. In this case, the solution is [-4,-2] since the polynomial is negative for all real numbers in this interval.

Absolute Value and Distance on the Real Number Line

The absolute value of a number can't be negative.

If a is a real number:

$$|a| = \begin{cases} a, if\ a \geq 0 \\ -a, if\ a \leq 0 \end{cases}$$

***Example 1.2f*: Find the absolute value of -8.**

If real number a = -8, then -8 ≤ 0. So, we use -a.

|-8| = -a = -(-8) = 8

Properties of Absolute Value

Let...
a and b be real numbers
n be a positive integer
k be a positive real number

Square root definition:
$|a| = \sqrt{a^2}$

Subadditivity:
$|a + b| \leq |a| + |b|$

Multiplication:
$|ab| = |a|\,|b|$

Quotient:
$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, b ≠ 0

Powers:
$|a^n| = |a|^n$

$-|a| \leq a \leq |a|$

$\lvert a\rvert \leq k$ if and only if	$-k \leq a \leq k$
$\lvert a\rvert < k$ if and only if	$-k < a < k$
$\lvert a\rvert \geq k$ if and only if	$a \leq -k$ or $a \geq k$
$\lvert a\rvert > k$ if and only if	$a < -k$ or $a > k$

Solving Absolute Value Inequalities

***Example 1.2g*: Solve the inequality $| x + 5 | < 10$**

From the property above we can see that $|a| < k$ if and only if $-k < a < k$

Therefore, we apply this to our inequality:

$-10 < x + 5 < 10$

$-10 - 5 < x + 5 - 5 < 10 - 5$ Subtract 5 from all parts

$-15 < x < 5$ Solution

The solution set is the single interval $(-15, 5)$.

***Example 1.2h*: Solve the inequality $| x + 6 | \geq 8$**

From the property above we can see that $|a| \geq k$ if and only if $a \leq -k$ or $a \geq k$

Therefore, we apply this to our inequality:

$x + 6 \leq -8$ or $x + 6 \geq 8$

For $x + 6 \leq -8$, $x \leq -14$

For $x + 6 \geq 8$, $x \geq 2$

The solution set is the union of the disjointed intervals $(-\infty, -14]$ and $[2, \infty)$.

Distance and Midpoint

Distance between two points with real numbers a and b:

$d = |a - b| = |b - a|$

***Example 1.2i*: Find the distance between -6 and 3.**

Let a = -6 and b = 3

$d = | a - b | = |-6 - 3| = |-9| = 9$

In both directions, -6 and 3 are 9 units from each other.

From a to b (going left to right), the directed distance would be 9. This is the directed distance from a to b. To compute, we use b - a. In this example, 3 - (-6) which would be 3 + 6 to get 9.

From b to a (going right to left), the directed distance would be -9. This is the directed distance from b to a. To compute, we use a - b. In this example, -6 - 3 to get -9.

The midpoint is the average value of endpoints a and b.

The midpoint of interval (a, b):

$$\frac{a + b}{2}$$

***Example 1.2j*: Find the midpoint of the interval [4, 16]**

Let a = 4 and b = 16

$$\frac{4 + 16}{2} = \frac{20}{2} = 10$$