Precalculus Review
Handout 1.3
Analytic Geometry

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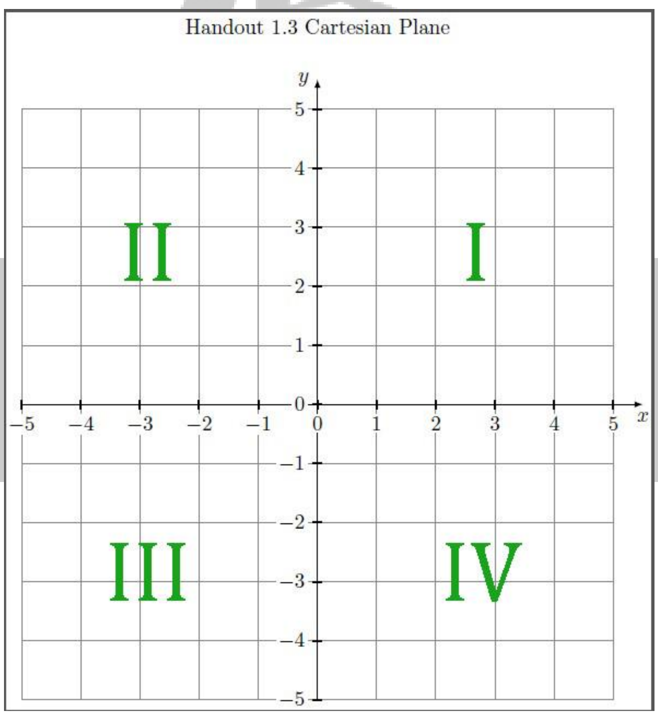
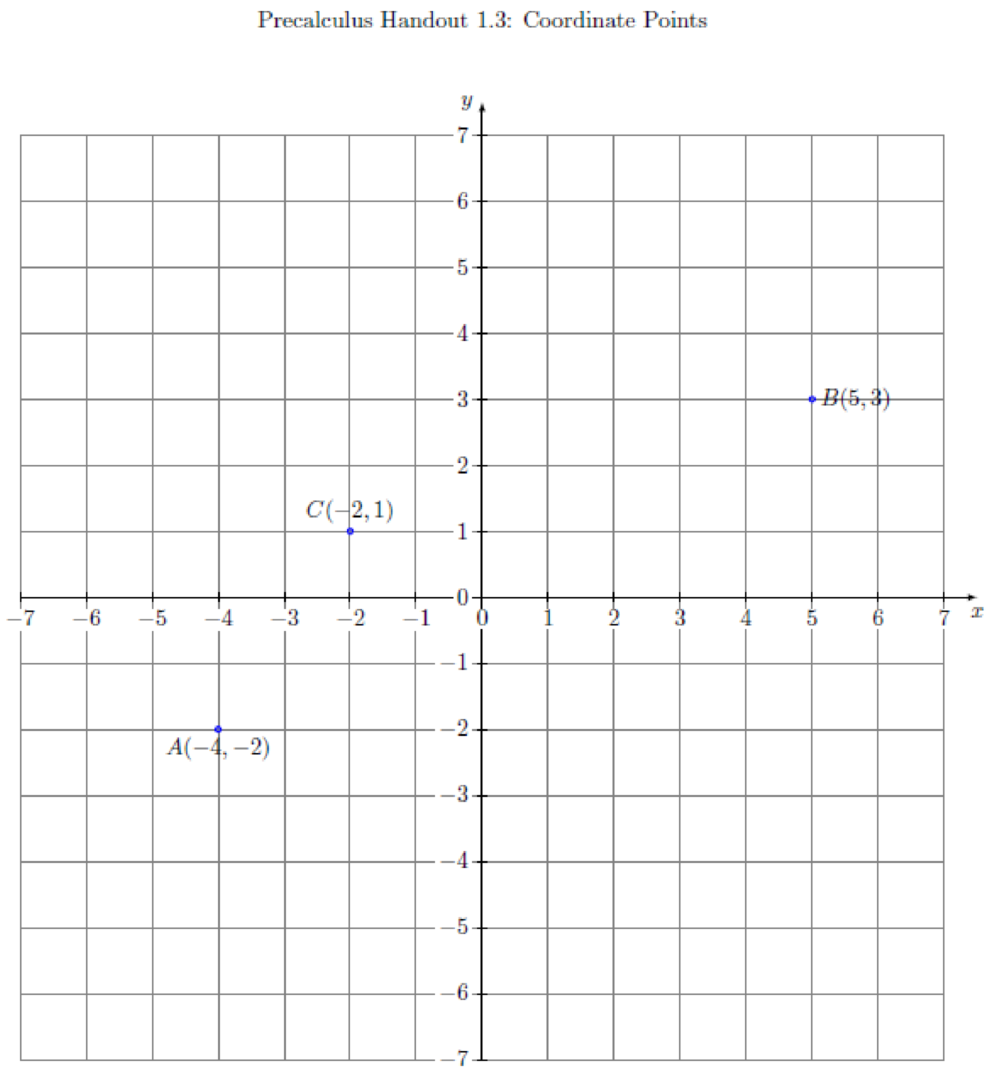
The Cartesian (Coordinate) Plane

A Cartesian (or coordinate) plane is formed when two real number lines intersect. The horizontal x-axis and the vertical y-axis intersect at right angles. The **origin** (0, 0) is the point of intersection for the x- and y- axes.

As a result, the plane is divided into four (4) quadrants.



Quadrant I starts at the upper right quadrant and rotates counterclockwise around the origin as it is numbered. The points in the coordinate plane are identified by an ordered pair (x, y) where x and y are real numbers. The x-coordinate is the directed distance of point *x* from the *y*-axis. The *y*-coordinate is the directed distance of point y from the *x*-axis.



Coordinate Points in a plane

The Distance Formula

In our previous handout, we discussed the distance between two points on the real number line. We are now going to continue further and find the distance between two points in the coordinate plane.

Let the first point be (x_1, y_1)

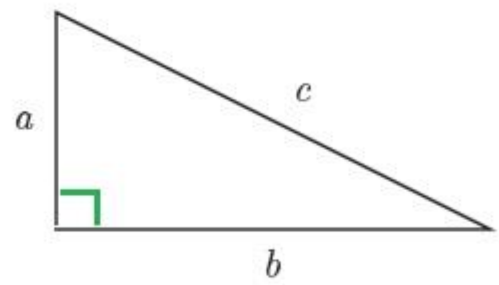
Let the second point be (x_2, y_2)

If you remember from Euclidean geometry:

The Pythagorean Theorem states that in a right triangle, the sum of the squares of the lengths of the two sides (a^2 and b^2 respectively) equals the square of the length of the hypotenuse (c^2).

Handout 1.3 Pythagorean Theorem

$$a^2 + b^2 = c^2$$



When we look at the two points in the coordinate plane, we can form a right triangle if the two points do not lie in the same vertical line ($x1 \neq x2$) or in the same horizontal line ($y1 \neq y2$) as each other.

By application of the Pythagorean Theorem,

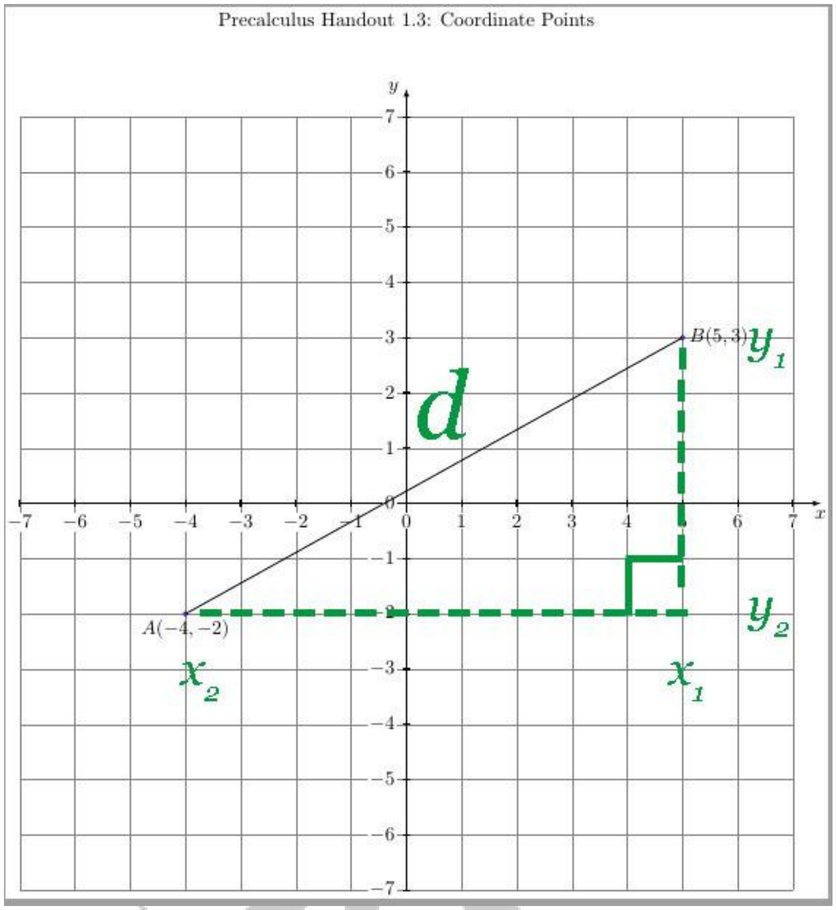
Let the vertical side (length a) be: $|y_2-y_1|^2$ equivalently $(y^2-y^1)^2$

Let the horizontal side (length b) be: $|x_2-x_1|^2$ equivalently $(x^2-x^1)^2$

The distance formula is given by:

$$\mathbf{d} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

where **d** is the distance between the two points (x_1, y_1) and (x_2, y_2)



Points A and B

Let's now apply this through an example.

Example 1.3a

Find the distance between the points (5, 3) and (-4, -2).

$\boldsymbol{d} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$
\begin{aligned}
\boldsymbol{d} &= \sqrt{(-4-5)^2 + (-2-3)^2} \\
&= \sqrt{(-9)^2 + (-5)^2} \\
&= \sqrt{81 + 25} \\
&= \sqrt{106} \\
&\approx 10.30
\end{aligned}
$$

If you switch around the points and make $(x_1, y_1) = (-4,-2)$ and $(x_2, y_2) = (5, 3)$, you will still get the same answer.

$$
\begin{aligned}
\boldsymbol{d} &= \sqrt{(5-(-4))^2 + (3-(-2))^2} \\
&= \sqrt{(9)^2 + (5)^2}
\end{aligned}
$$

$= \sqrt{81 + 25}$
$= \sqrt{106}$
≈ 10.30

Now let's apply the distance formula for other situations.

Example 1.3b

Let x be an unknown coordinate in point B.

If the distance between point A (3, 4) and point B (x, 8) is 4, find x.

4	$= \sqrt{(x-3)^2 + (8-4)^2}$
16	$= (x^2 - 6x + 9) + 16$
0	$= (x^2 - 6x + 9) + 16 - 16$
0	$= (x^2 - 6x + 9) + 0$
0	$= (x - 3)^2$

The solution is $x = 3$. The coordinate for point B is (3, 8). When we use the distance formula, we can verify this solution.

d	$= \sqrt{(3-3)^2 + (8-4)^2}$
	$= \sqrt{(0)^2 + (4)^2}$
	$= \sqrt{16}$
	$= 4$

<u>The Midpoint Formula</u>

The midpoint of a line between two points is calculated with the formula:

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

where our midpoint is the average of x_1 and x_2 and the average of y_1 and y_2.

Example 1.3c

Find the midpoint of the line segment connecting the points (2, 7) and (6, -5).

$$\left(\frac{2+6}{2}, \frac{7+(-5)}{2}\right) = \left(\frac{8}{2}, \frac{2}{2}\right)$$

The midpoint of the line segment is (4, 1).

Equations of a Circle

We can also extend the applications of the distance formula involving the equations of a circle.

The set of all points equidistant from the fixed center point on a plane form a circle.

The radius of a circle, $\boldsymbol{r}$, is the distance between the center point $(\boldsymbol{h}, \boldsymbol{k})$ and a point found on the circle $(\boldsymbol{x}, \boldsymbol{y})$.

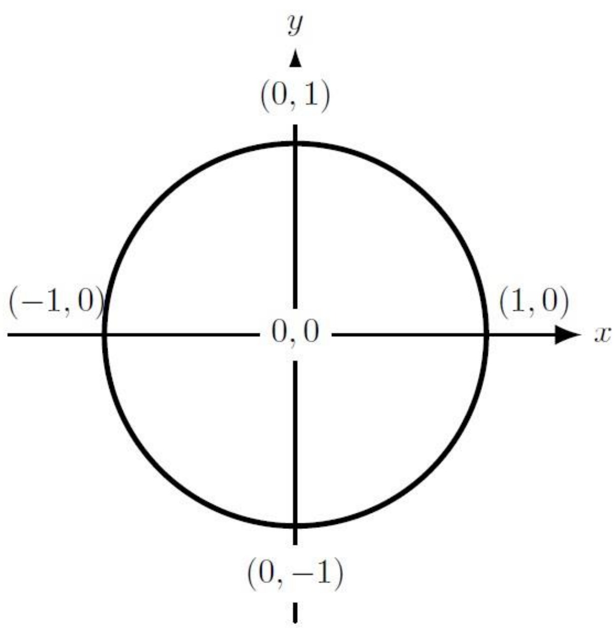
The distance between a point on a circle and the center is noted by

$$\sqrt{(\boldsymbol{x}-\boldsymbol{h})^2+(\boldsymbol{y}-\boldsymbol{k})^2} = \mathbf{r}$$

The standard form of the equation of a circle can be obtained by squaring both sides of the equation.

$$(\boldsymbol{x} - \boldsymbol{h})^2 + (\boldsymbol{y} - \boldsymbol{k})^2 = \boldsymbol{r}^2$$

A circle with a radius of 1 (r = 1) is referred to as the **unit circle**.



The unit circle
Center is at the origin $(h, k) = (0, 0)$
Radius $r = 1$

We will now look at an example where the center does not lie on the origin.

Example 1.3d

The point (6, 8) is located on a circle whose center is at (4, 4). Write the equation of this circle in standard form.

Step 1: First we will need to find the radius of the circle using the distance formula.

$\sqrt{(6-4)^2+(8-4)^2} = \mathbf{r}$
$\sqrt{(2)^2+(4)^2} = \mathbf{r}$

$\mathbf{r} = \sqrt{20}$
(You can also write it as $2\sqrt{5}$ if you prefer, but in any case we will get the same answer in the end)

Step 2: We write the equation of the circle by squaring both sides of the equation. We use the given information of the center (h, k) and the radius, r, to write the equation in standard form.

The standard form of the equation of the circle is noted by

$\boldsymbol{(x - h)^2 + (y - k)^2 = r^2}$

The standard form of a circle with a center at (4, 4) and a radius of $\sqrt{20}$ is

$(x - 4)^2 + (y - 4)^2 = (\sqrt{20})^2$

$\boldsymbol{(x - 4)^2 + (y - 4)^2 = 20}$ **The standard form of the equation**

<u>General Form of the Equation of a Circle</u>

The general form of the equation of a circle is noted by

$$\mathbf{A}x^2 + \mathbf{A}y^2 + \mathbf{D}x + \mathbf{E}y + \mathbf{F} = 0, \quad \mathbf{A} \neq 0$$

We already learned the standard form of the equation of a circle. We will use a method called **completing the square** to convert an equation written in general form into the standard form:

$$(x - h)^2 + (y - k)^2 = p$$

If $p < 0$, equation does not have a graph
If $p = 0$, the graph will be the single point (h, k)
If $p > 0$, the graph of the equation will be a circle

Let's look at an example to examine the method of completing the square.

Example 1.3e

Given the general form of the equation of the circle:

$4x^2 + 4y^2 - 4x + 2y - 1 = 0$

Write the equation in standard form and sketch its graph.

Step 1: If the coefficients for x^2 and y^2 are not 1, divide by the coefficients of x^2 and y^2. If the coefficients are 1 (as in x^2 and y^2), you can proceed to step 2.

In our example, the coefficients are 4. We will need to divide everything by 4. Once we do that we will have:

$x^2 + y^2 - x + \frac{1}{2}y - \frac{1}{4} = 0$

Step 2: Group the terms together. Put the x terms together and the y terms together. Bring the number without an x or y to the right side of the equation.

$x^2 - x + y^2 + \frac{1}{2}y = \frac{1}{4}$

Once the terms are grouped, we put the grouped x terms in parentheses and the grouped y terms in parentheses. We then add a plus sign to begin completing the square.

$(x^2 - x + \quad) + (y^2 + \frac{1}{2}y + \quad) = \frac{1}{4}$

Step 3: Complete the square by taking half of the coefficient of x and half of the coefficient of y and then squaring it. We then add them to both sides of the equation.

For x:

In our example, the coefficient of x is -1. If we divide it by 2, we get $-\frac{1}{2}$. We then square the $-\frac{1}{2}$ to complete the square. $(-\frac{1}{2})^2 = \frac{1}{4}$

For y:

The coefficient is $\frac{1}{2}$. If we divide by 2, we get $\frac{1}{4}$. We then square it: $(\frac{1}{4})^2 = \frac{1}{16}$

Remember to also add $\frac{1}{4}$ and $\frac{1}{16}$ to the right side of the equation!

Now we get:

$$(x^2 - x + \frac{1}{4}) + (y^2 + \frac{1}{2}y + \frac{1}{16}) = \frac{1}{4} + \frac{1}{4} + \frac{1}{16}$$

For the left side, we simplify the equation and write it in standard form:

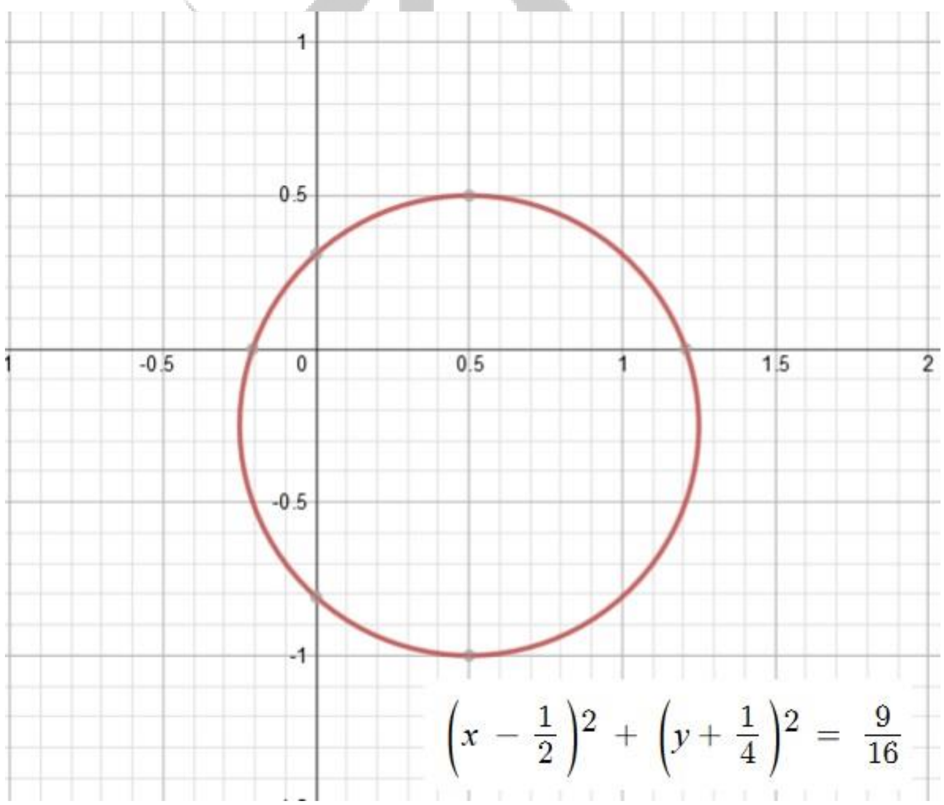
$$(x - \frac{1}{2})^2 + (y + \frac{1}{4})^2$$

For the right side, we add them using common denominators. Since 16 is a common denominator for $\frac{1}{4} + \frac{1}{16}$, we just convert and add. $\frac{1}{4} = \frac{4}{16}$. Our p = $\mathbf{\frac{9}{16}}$.

Now we put it together to write it in standard form.

$$\boldsymbol{(x - \frac{1}{2})^2 + (y + \frac{1}{4})^2 = \frac{9}{16}}$$

Since p > 0, our graph is a circle.



Graph courtesy of Desmos