# Precalculus Review
# Handout 1.5
# Trigonometric Functions Part 2
# Applications of the Trigonometry of Right Triangles

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In our previous handout, we defined the trigonometric functions as they related to right triangles. The concepts here deal with right triangles. Later on, we will use different methods for other triangles such as the law of sines.

As we start applying trigonometric concepts, we will see their uses throughout many real world examples such as calculating distance and speed.

## The Cofunction Identities

When we took a look at the special right-triangles (45-45-90 and the 30-60-90), we noted that the other two acute angles are **complementary** (meaning that adding them together will equal 90 degrees). In the 45-45-90 isosceles triangle, the two acute angles are $45^\circ + 45^\circ$ which equal to $90^\circ$. In the 30-60-90, the two acute angles are $30^\circ + 60^\circ$ which also equal to $90^\circ$.

The confunction identities show the relationship between a trigonometric function and its cofunction.

To make it easier to learn, think of the trigonometric function and add “co-” in front of it (“co” for **co**mplementary).

sine → **co**sine

secant → **co**secant

tangent → **co**tangent

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$$

$$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$$

Similarly, if you want to find cos θ, csc θ, or cot θ:

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

$$\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$$

$$\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$$

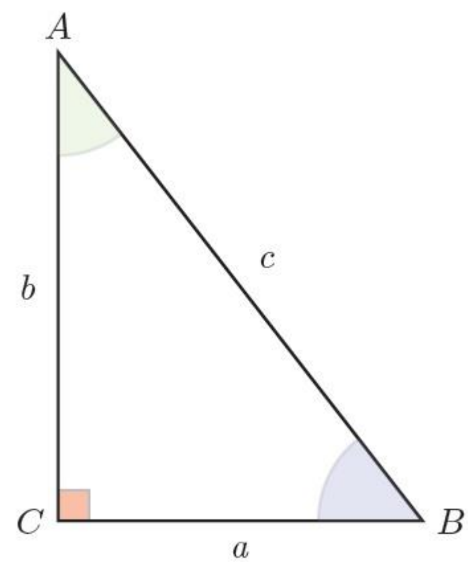
****You can also exchange $\frac{\pi}{2}$ with 90˚ since we learned that $\frac{\pi}{2}$ = 90˚.** For example,

$$\sin\theta = \cos(\,90^\circ - \theta)$$

Please keep in mind to not confuse these as the reciprocals of each other! The reciprocal identities were covered in our previous trigonometry handout (Handout 1.4).

Don't worry if the cofunction identities look overwhelming right now, we will go over them step by step. We will begin by looking at a diagram and see how the cofunction identities apply.

# Handout 1.5 Cofunction Identities for Right Triangles




We learned that $\sin\theta = \frac{opposite}{hypoteneuse}$ which can now be applied to our triangle:

$$\mathbf{\sin A = \frac{\boldsymbol{a}}{\boldsymbol{c}} = \cos B}$$

and $\cos\theta = \frac{adjacent}{hypoteneuse}$

$$\mathbf{\cos A = \frac{\boldsymbol{b}}{\boldsymbol{c}} = \sin B}$$

**Whenever angles A and B are complementary (that is, they add up to 90 degrees) and the trigonometric function is a complement of a given angle (e.g. sin A = cos B), they are cofunctions of each other.**

To see sin A = cos B in greater detail:

$$\mathbf{\sin A = \frac{\boldsymbol{opposite}}{\boldsymbol{hypoteneuse}} = \frac{\boldsymbol{a}}{\boldsymbol{c}}}$$

$$\mathbf{\cos B = \frac{\boldsymbol{adjacent}}{\boldsymbol{hypoteneuse}} = \frac{\boldsymbol{a}}{\boldsymbol{c}}}$$

Thus, we get **sin A = cos B**.

Now let's put in measurements for the angles in the above example.

Let angle A = 60˚. Since angles A and B are complementary, angle B = 30˚

We can use our cofunction identity:

$$\sin \theta = \cos(90° - \theta)$$

$$\sin 60° = \cos(90° - 60°)$$

$$\sin 60° = \cos 30°$$



Similarly:

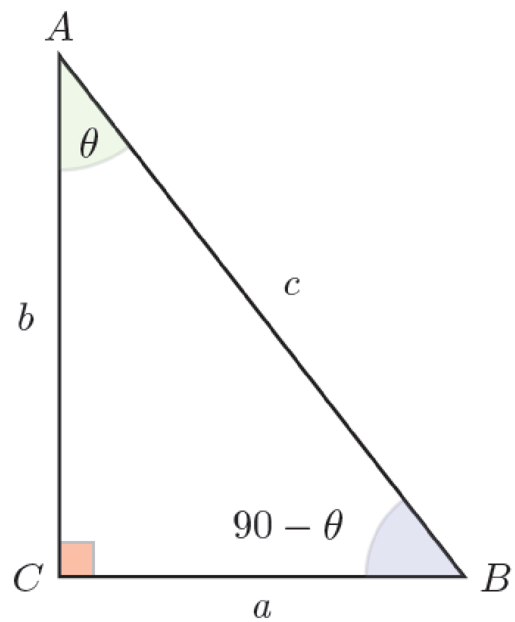
$$\sin \theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\sin \frac{\pi}{3} = \cos\left(\frac{\pi}{2} - \frac{\pi}{3}\right)$$

$$\sin \frac{\pi}{3} = \cos \frac{\pi}{6}$$

Let's take another look at our triangle using the cofunction identities and let angle A = θ. Since angle B is complementary to angle A, it measures 90˚ - θ.

# Handout 1.5 Cofunction Identities for Right Triangles




We can use these identities for a wide range of applications. For example, if we are given a trigonometric value (such as sin 72˚) and we want to verify the value of the cofunction of the complementary angle, we can use the confunction identities.

*<u>Example 1.5a</u>*

Verify the cofunctional relationships given the following information:

a. $\sin 72^\circ$
b. $\tan 58^\circ$

c. $\sec \frac{3\pi}{8}$

d. $\sin \frac{2\pi}{5}$



**Step 1: Identify the confuctional relationships.**

Let's take a look at the first one: $\sin 72^\circ$

We learned that the cofunction of sine is cosine.

**Step 2: Use the cofunction identity to find the value of the cofunction.**

$$\boldsymbol{sin\,\theta} = \mathbf{cos}(\mathbf{90^\circ} - \boldsymbol{\theta})$$

$$\boldsymbol{sin}\ \mathbf{72°} = \mathbf{cos}(\mathbf{90°} - \mathbf{72°})$$

$$\boldsymbol{sin}\ \mathbf{72°} = \mathbf{cos}\ \mathbf{18°}$$

**Step 3: Verify your answer.**

Using a calculator we can find the values for the given trigonometric functions.

$\sin 72^\circ \approx 0.9510565$

$\cos 18^\circ \approx 0.9510565$

Now let's do **c. (sec = $\frac{3\pi}{8}$)** since it is given in radians instead of degrees.

**Step 1: Identify the cofunctional relationships.**

We learned that the cofunction of secant is cosecant.

**Step 2: Use the cofunction identity to find the value of the cofunction.**

$$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$$

$$\sec\frac{3\pi}{8} = \csc\left(\frac{\pi}{2} - \frac{3\pi}{8}\right)$$

$$\sec\frac{3\pi}{8} = \csc\frac{\pi}{8}$$

**Step 3: Verify your answer.**

$\sec\frac{3\pi}{8} \approx 2.6131$

$\csc\frac{\pi}{8} \approx 2.6131$

****Don't forget to keep the modes in mind when using the calculator! If you are verifying degrees, use DEG mode. If you are verifying radians, use RAD mode.**

For the rest of them, we can follow the same steps.

b. $\tan 58^\circ = \cot 32^\circ$

d. $\sin\frac{2\pi}{5} = \cos\frac{\pi}{10}$

## Finding Acute Angle Value

Suppose we are given the information sin θ = 0.7547 and we are asked to find the value of θ to the nearest degree in the interval between 0 and 90 degrees.

Let's take a look at how to find θ step by step. (After this example, we will do an example finding θ using the right triangle definitions)

**Step 1: Organize your information.**

sin θ = 0.7547

θ = ?

**Step 2: Use the inverse trigonometric function to find the acute angle associated with our original trigonometric function.**

We will examine inverse trigonometric functions in depth later on, but will touch upon them briefly here.

From our previous handout, we looked at the special right triangles. We learned the values of the measures of the angles in Quadrant I. For example, if sin θ = $\frac{\sqrt{2}}{2}$, we learned that θ = 45˚ or $\frac{\pi}{4}$ in the 45-45-90 isosceles triangle since according to our right triangle definitions, sin θ = $\frac{opposite}{hypoteneuse}$.

Similarly, if tan θ = $\sqrt{3}$, we learned that θ = 60˚ or $\frac{\pi}{3}$ in the 30-60-90 triangle since according to our right triangle definitions, tan θ = $\frac{opposite}{adjacent}$ .

To find an unknown value for our angle (θ, x, or whichever variable is used), we will utilize the inverse trigonometric functions (arcsine, arccosine, arctangent, etc.).

For our example we are given sin θ = 0.7547. Our next step is to take the arcsin of 0.7547.

Using a calculator, set it to radian mode (**RAD**) and locate arcsin (it should look like **$\sin^{-1}$**). Please be careful to check that it is not in degree mode (**DEG**) since you will get a completely different answer.

Input $\sin^{-1}(0.7547)$. $\sin^{-1}(0.7547) \approx 0.8551967$ radians

**Step 3: Convert your answer to degrees.**

$$\frac{0.8551967\ \cancel{\text{radians}}}{} \cdot \frac{180 \text{ degrees}}{\pi\ \cancel{\text{radians}}}$$

$= 48.999$ degrees $\approx$ **$49^\circ$**

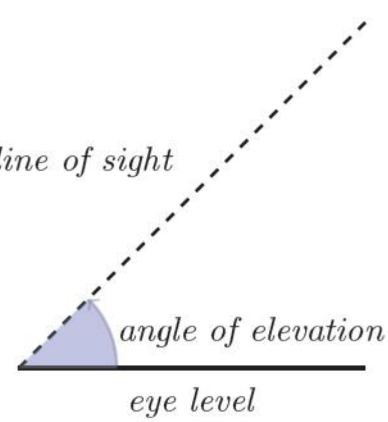


**Angle of Depression and Angle of Elevation**

We are now going to discuss two important concepts frequently encountered in trigonometric applications: angle of depression and angle of elevation.

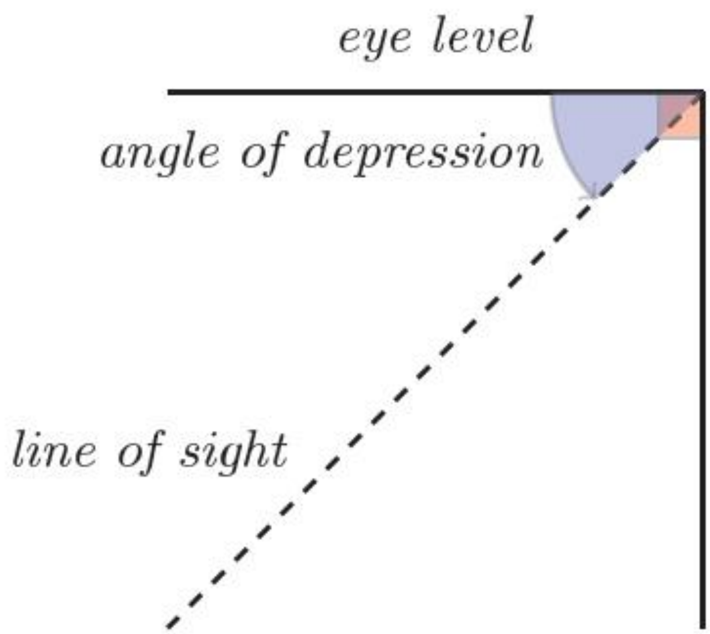
**Angle of elevation** is when an individual's eyes shift upward from the horizontal line (eye level) to focus on an object in a higher position. For example, if an individual's eyes focus on the top of a flagpole or tree.

Handout 1.5 Angle of Elevation



**Angle of depression** is when an individual's eyes shift downward from their horizontal line of sight (eye level) to focus on an object in a lower position. For example, if a standing individual focuses his or her eyes on a duck or flower on the ground.

Handout 1.5 Angle of Depression




We will be referring to these two concepts as we look at the next few examples.

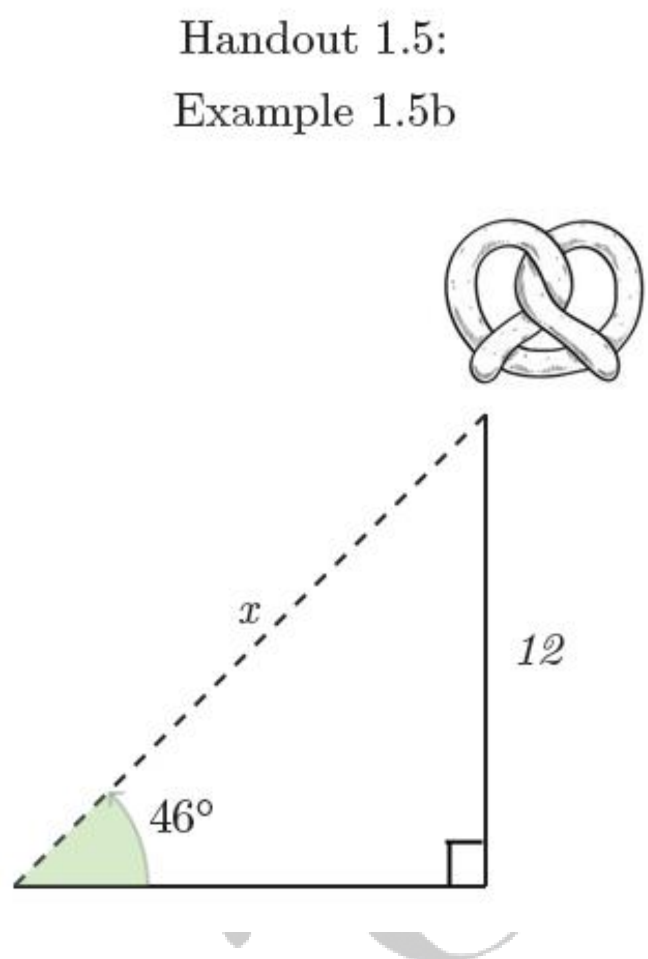


*Example 1.5b*

Jason is at the first floor finishing up holiday shopping at the mall. He gets hungry and sees a pretzel stand on the second floor. The angle of elevation to the pretzel stand from Jason is 46˚. The distance from the first floor to the second floor is 12 meters. How far is Jason from the pretzel stand?

**Step 1: Construct a diagram and label the known information from the problem.**

One of the most important steps that I told my students and anyone attempting to solve mathematical problems is to always draw out a diagram. It serves as a useful reference guide and can help you organize your information. Of course, you don't have to be an artist or have to draw an elaborate diagram, but as long as it is understood and labeled correctly, you should be fine.




**Step 2: Use the given information and right triangle definitions to solve the problem.**



When we look at our diagram, we are given the information opposite the 46 degree angle (12 meters) and our unknown distance (x) which forms the hypotenuse.

As you can recall, $\sin\theta = \frac{opposite}{hypotenuse}$.

We just put our information from the diagram into the right triangle definition.

$$\sin 46^\circ = \frac{12}{x}$$

Using algebraic manipulation to isolate the x:

$$x = \frac{\mathbf{12}}{\mathbf{\sin 46^\circ}}$$

$$\boldsymbol{x \approx 16.68 \text{ meters}}$$

**Jason's distance from the pretzel stand is around 16.68 meters.**

<u>*Example 1.5c*</u>

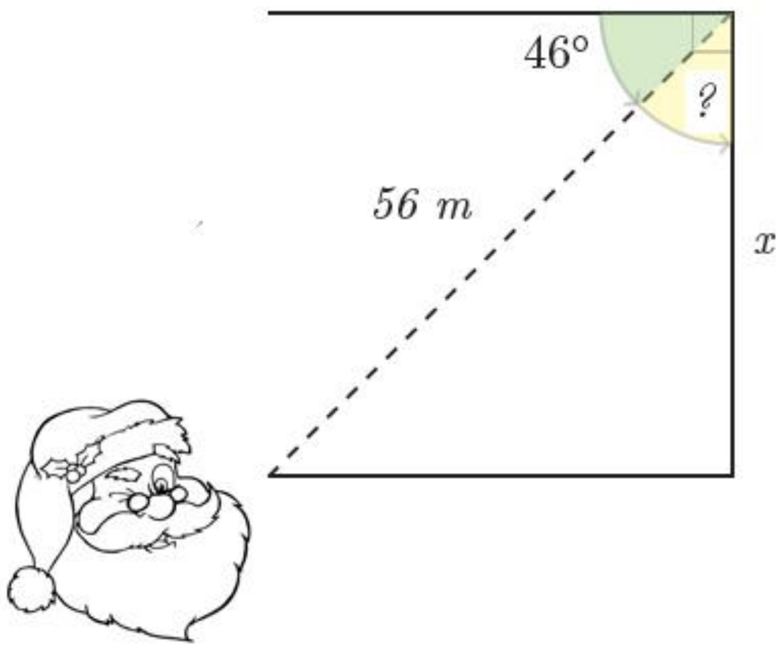
Sarah and her little brother Mike are walking on the fourth floor of the mall. Mike sees Santa Claus on the first floor and tells his sister that he would like to visit him. If the distance between Mike and Santa Claus is 56 meters and the angle of depression from Mike to Santa Claus is 64˚, find the vertical distance between the first and fourth floors.

This problem will be a lot more involved. We will use our knowledge of complementary angles along with our methods from Example 1.5b.

**Step 1: Construct a diagram and label the known information from the problem.**

Similar to our previous problem, we create a diagram and label the information provided to us.

# Handout 1.5 Angle of Depression




**Step 2: Use the given information and find the complement to the angle of depression.**

The distance between Mike and Santa Claus forms the hypotenuse (56 meters). The angle of depression from Mike to Santa Claus is 64˚. To proceed, we have to create a right triangle. Let ***x*** equal the vertical distance between the first and fourth floors.

Although we have the two sides, the angle of depression is not in our triangle. However, we learned that the complementary angle to our angle of depression is 90˚ - 64˚ = 26˚. The complementary angle (yellow angle denoted as ?) in our diagram is equal to 26˚.

We now have all of the information to solve the problem.

**Step 3: Use the right triangle definition to find the unknown side.**

The unknown side in our problem, ***x***, is the vertical distance between the first and fourth floors. From our diagram, we can see that the information gives us $\cos\theta = \frac{adjacent}{hypotenuse}$.

The unknown side is adjacent and the hypotenuse is the distance between Mike and Santa Claus.

$$\cos 26° = \frac{x}{56}$$

$$x = 56 \cos 26°$$

$$x \approx 50.33 \text{ meters}$$

**The vertical distance between the first and fourth floors measures 50.33 meters.**



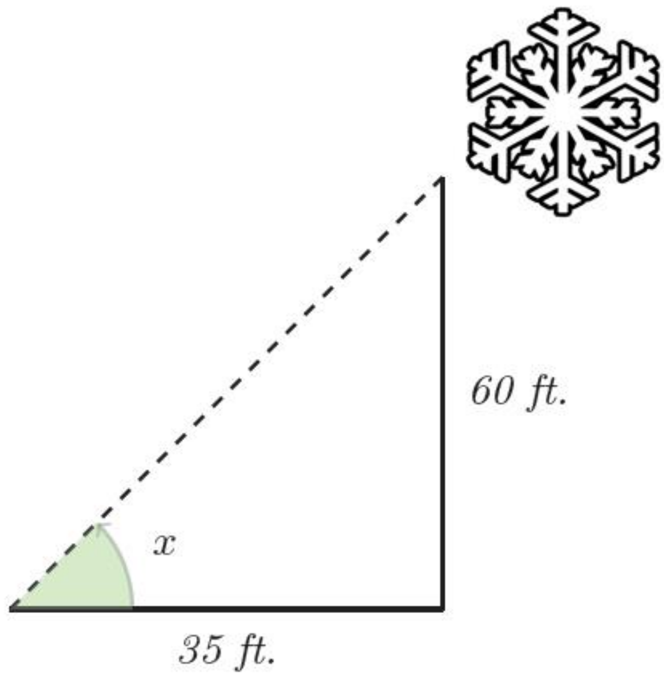
<u>*Example 1.5d*</u>

A tourist wants to take a picture of a giant lighted snowflake display attached to a building.  If the tourist is 35 feet away from the base of the building and the snowflake display is placed at a height of 60 feet, approximate the angle where the tourist's camera should be pointed to capture the snowflake display.

**Step 1: Construct a diagram and label the known information from the problem.**

In our diagram, we are finding the angle of elevation (the angle formed by the camera and eye level).

Handout 1.5:
Example 1.5d




**Step 2: Use the right triangle definition to find the appropriate trigonometric function.**

Since the height of the snowflake display is opposite of our angle and the tourist's position forms the adjacent side, we use tangent.

$$\tan\theta = \frac{opposite}{adjacent}$$

$$\tan\theta = \frac{60}{35}$$

**Step 3: Use the inverse trigonometric function to find the value of the acute angle it.**

If you remember from the acute angle value section earlier in this handout, we used the inverse trigonometric function to the value of the acute angle that it is associated with. In our case, we will use arctangent:

$$\tan^{-1}\left(\frac{60}{35}\right)$$

or

$$\arctan\left(\frac{60}{35}\right)$$

$$\approx 1.04272 \text{ radians}$$

Step 4: Convert the radian measure into degrees.

| 1.04272 ~~radians~~ | 180 degrees |
|---|---|
| | $\boldsymbol{\pi}$ ~~radians~~ |

= 59.743 degrees ≈ **60°**

**The camera must be pointed around 60° in order for the tourist to capture the snowflake display.**

## Area of a Triangle

From our earlier studies in mathematics, we learned the formula for the area of a triangle:

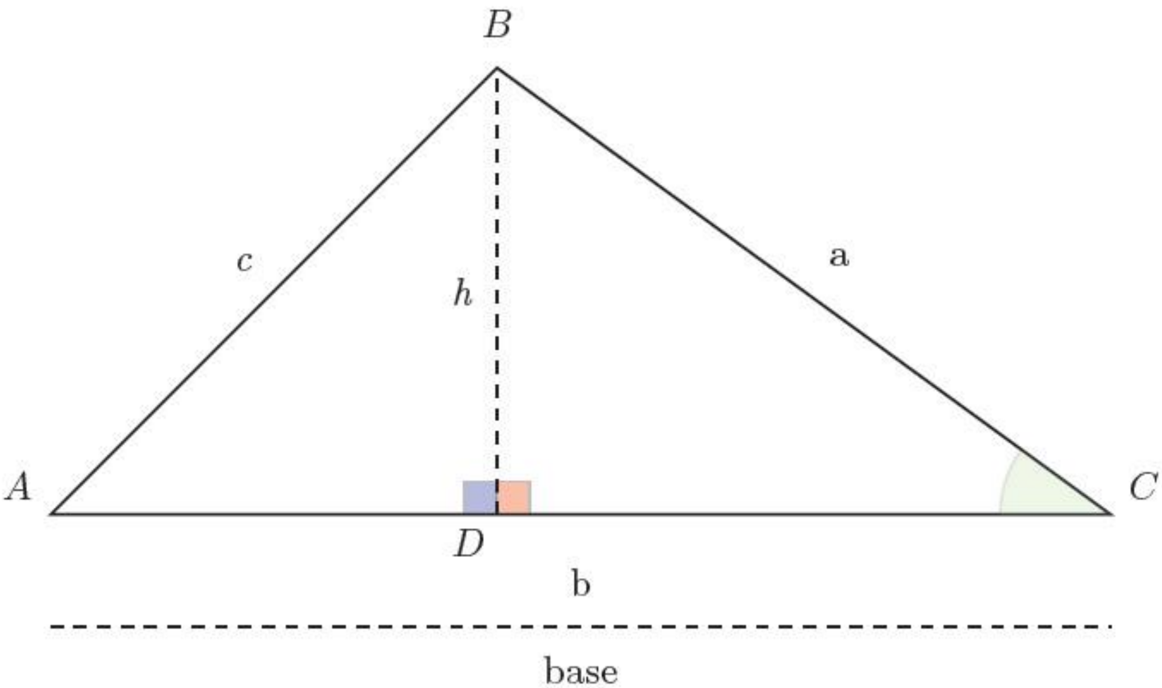
$$\boldsymbol{A = \frac{1}{2}\, bh}$$

We will now extend this formula to its trigonometric form:

$$\boldsymbol{A = \frac{1}{2} ab \sin C}$$

Let's first take a look at our triangle using the first form of the formula for the area of a triangle.

# Handout 1.5 Area of a Triangle




In our diagram:

***b*** represents the side opposite of ∠B (∠ is the symbol for angle) and is the base for our triangle

***h*** represents the height

**∠C** is highlighted in green

The red and blue highlights represent right angles (90˚).



Now when we look at the trigonometric form of the triangle, let's derive how we got the formula from our given diagram:

$$\boldsymbol{A = \frac{1}{2}ab\sin C}$$

We know our base is ***b***. Now we will find ***h***.

In our formula we have to relate height, $\boldsymbol{h}$, to sin ∠C. Since we have established base, $\boldsymbol{b}$, we need to figure out where the **a sin C** comes into the equation. From the right triangle definition, $\mathbf{sin\ \theta} = \frac{\boldsymbol{opposite}}{\boldsymbol{hypotenuse}}$. We then substitute ∠C for θ to get:

$$\sin C = \frac{opposite}{hypotenuse} = \frac{h}{a}$$

After we do that, we just need to solve for $\boldsymbol{h}$:

$$\sin C = \frac{h}{a}$$

We multiply both sides by a to cancel out the $\boldsymbol{a}$ on the right side:

$$(a)\sin C = \frac{h}{a}(a)$$

We then get our h: $\boldsymbol{h = a \sin C}$

When we substitute $\boldsymbol{b}$ and $\boldsymbol{h}$ into our first formula:

$$A = \frac{1}{2}bh$$
$$A = \frac{1}{2}(b)(a \sin C)$$
$$A = \frac{1}{2}ab \sin C$$

With this formula, we can find the area of a triangle when the height (altitude) is unknown. If we are given the two sides (a and b) and the value of ∠ C, we can use this formula to find the area.

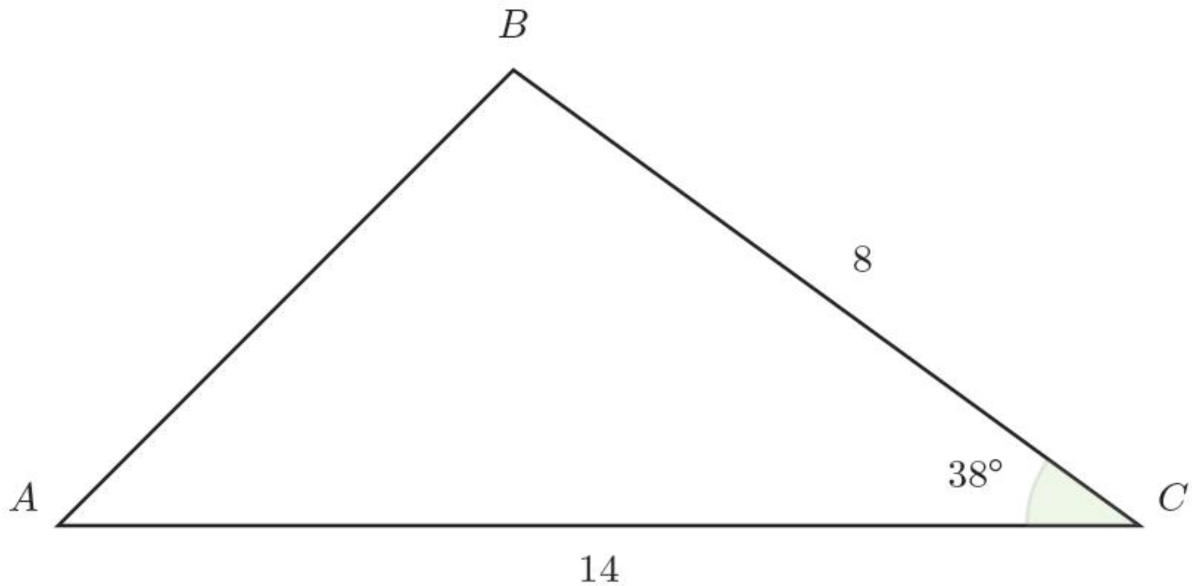
Let's take a look at an example to see it in action.

***<u>Example 1.5e</u>***

Find the area of a triangle with two sides measuring 8 cm. and 14 cm. and ∠ C measuring 38°.

**Step 1: Construct a diagram and label the known information.**

Handout 1.5: Example 1.5e




**Step 2: Use the trigonometric formula for the area of a triangle.**

$$A = \frac{1}{2}ab\sin C$$

$$A = \frac{1}{2}(8)(14)\sin 38°$$

$$A = 56\sin 38°$$



$$\mathbf{A \approx 34.48\ cm.}$$

**<u>Area of a Sector</u>**

In our previous handout, we took a look at the arc length of a sector, $\mathbf{s = r\theta}$. In this section we will now cover the area of a sector. We will see these applications when we try to find a shaded area with polygons inscribed in another polygon (e.g. a square or pentagon inscribed inside a circle).

The formula for the area of a sector is given by:

$$\boldsymbol{A = \frac{1}{2}\, r^2\theta}$$

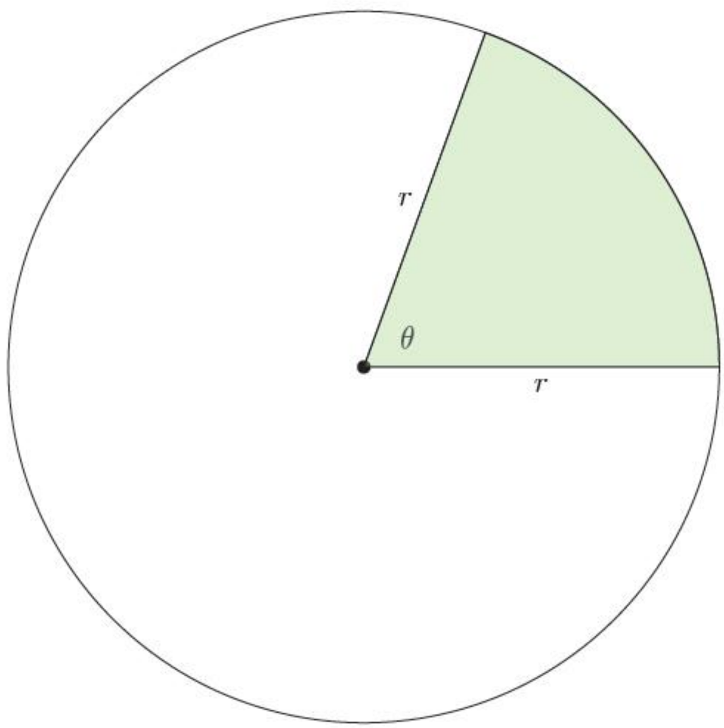
where **θ is measured in radians**.

If θ is measured in degrees, the formula for the area of a sector is given by:

$$\boldsymbol{A = \frac{\theta^\circ}{360^\circ}\pi r^2}$$

In most cases, θ is usually measured in radians. As you can see, it's easier to use the formula when θ is in radians versus degrees.






In our diagram above, just replace θ with θ˚ if the central angle is measured in degrees.

To see how the formulas relate to the area of a complete circle, we first look at the formula of the area of a circle. The area of a circle is $\mathbf{A = \boldsymbol{\pi r^2}}$.

As we can see from the diagram, the area of the sector (the green shaded region) is a piece of the complete circle. Since it is a piece of our complete circle, we can express it as a fraction (similar to examples with pieces of pie, pizza, or cake; if an entire pizza pie consists of eight slices and you eat one, you took $\frac{1}{8}$ of the entire pizza pie). Similarly, the shaded region is what we focus on.

Since the measure of the central angle of a complete circle is 360˚ (in degrees) or $2\boldsymbol{\pi}$ ( in radians), we can express the area of the sector (the shaded region) as a piece of it.

The sector will be expressed as $\frac{\boldsymbol{\theta}}{\mathbf{2}\boldsymbol{\pi}}$ or $\frac{\boldsymbol{\theta}^\circ}{\mathbf{360}^\circ}$. Since the area of a complete circle is $\mathbf{A} = \boldsymbol{\pi r^2}$, we include the area of a sector as a part of the complete circle and then express it as:

$$A = \frac{\theta}{2\pi}\pi r^2$$

We can then separate the $\frac{1}{2}$ and cancel out the $\boldsymbol{\pi}$ as we multiply and simplify:

$$A = \left(\frac{1}{2}\right)\left(\frac{\theta}{\pi}\right)\pi r^2$$

$$\boldsymbol{A = \frac{1}{2}r^2\theta}$$

That's how we get our formula for the area of a sector!

Similarly, if your θ is measured in degrees (θ˚), you use $\frac{\boldsymbol{\theta}^\circ}{\mathbf{360}^\circ}$ instead:

$$\boldsymbol{A = \frac{\theta^\circ}{360^\circ}\,\pi r^2}$$

As you can see, the $\boldsymbol{\pi}$ will make it a little more complicated when computing which is why many textbooks or exercises prefer to use the radian version above (usually, but not always). However, the process is the same for both when finding the area of a sector.

***Example 1.5f***

A regular hexagon is inscribed inside a circle with a radius of 4 centimeters. Find the area of the shaded region located outside of the hexagon and inside the circle.

We will use two methods: (1) subtracting the area of the hexagon from the area of the circle; (2) using the formula for the area of a sector and subtracting the area of the triangle from the sector.

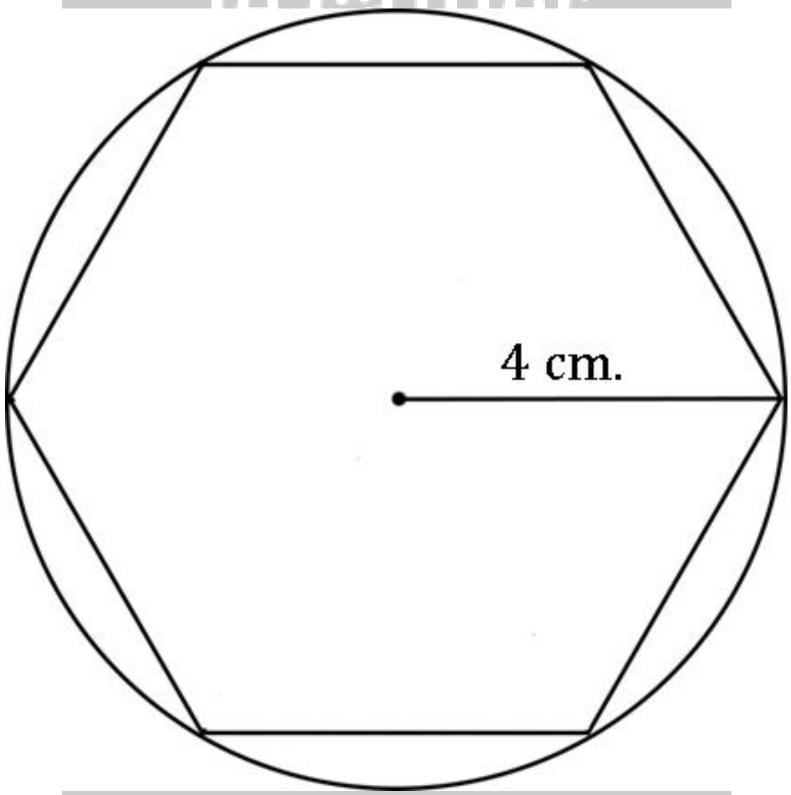
We will begin with number 1 first.

****A regular polygon means that all of the sides and angles are of equal length.**

## <u>Method 1</u>

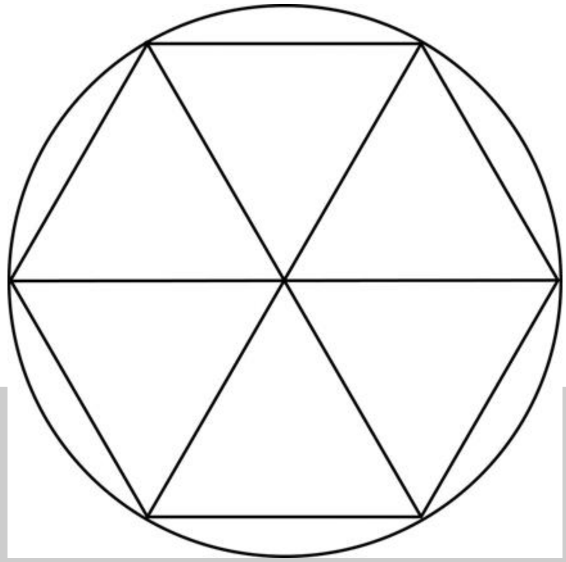
**Step 1: Construct the diagram and label the known information.**

The radius is 4 cm.

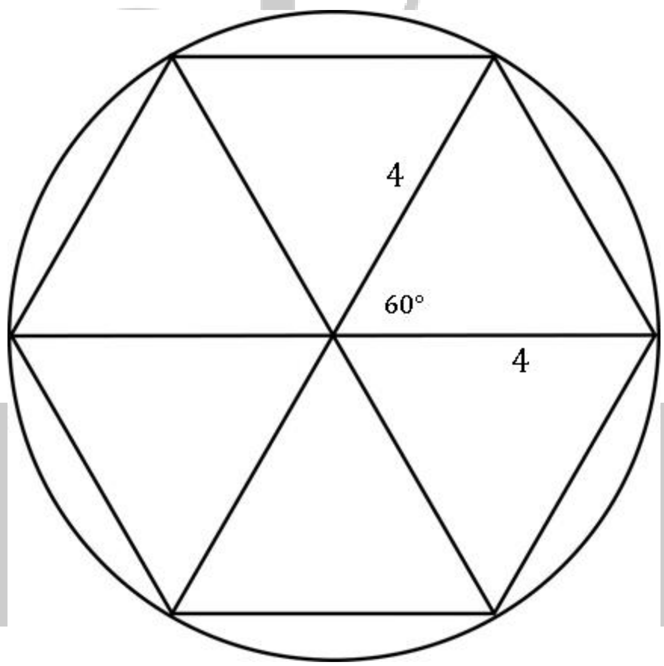



**Step 2: Use the formula for the area of a triangle.**

When we look at our original figure, we can divide the hexagon into six equilateral triangles. All of the angles within the triangle measure 60˚ (since the total measure of angles in a triangle is 180˚).



Since we are given the radius and we have a regular hexagon, we can note that the sides are equal.




$$A = \frac{1}{2}ab\sin C$$

$$A = \frac{1}{2}(4)(4)\sin 60°$$

$$A = 8\sin 60°$$

$A$ = 6.928 cm

The given answer is the area for **one** triangle. Since we have 6 triangles, we multiply that by 6 to get the area of the entire hexagon.

**Total Area = (6)(6.928) = 41.568 cm**

**Step 3: Use the formula for the area of a circle.**

The area of a circle is $A = \pi r^2$.

Our radius is 4 cm. Therefore, $A = \pi(4)^2$. **A ≈ 50.26548 cm**

**Step 4: Subtract the area of the hexagon from the area of the circle.**

When we subtract the area of the hexagon from the area of the entire circle, we get the area of the shaded region.

Area of the shaded region = Area of the entire circle - Area of the hexagon

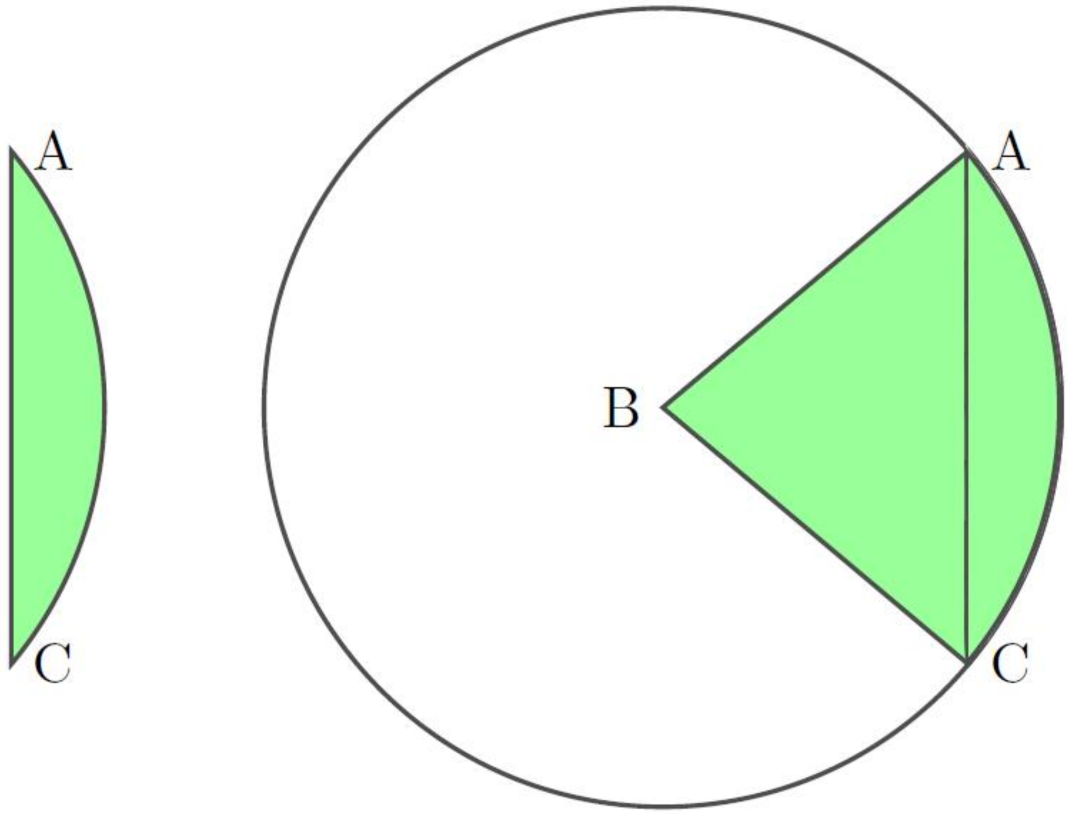
A = 50.265 - 41.568

A ≈ 8.697 cm. ≈ 8.7 cm

**The area of the shaded region is around 8.7 cm.**

**Method 2**

In this method, we will apply what we learned in this section.




We need to find the area of the shaded region (left).

**Step 1: Find the area of the entire sector.**

In our diagram, sector ABC includes our triangle and the shaded region. From the previous method, we learned that the triangle is an equilateral triangle with all angles measuring 60˚.

Since our θ is measured in degrees, we will use the degree version of the formula for the area of a sector.

$$\boldsymbol{A = \frac{\theta^\circ}{360^\circ}\pi r^2}$$

We know that the angle measures 60˚ and the radius measures 4 cm.

$$A = \frac{60^\circ}{360^\circ}\pi(4)^2$$

$$= \frac{1}{6}\,\pi(16)$$

$$= \frac{16}{6}\,\pi$$

$$= \frac{8}{3}\,\pi$$

$$\approx 8.3775$$

The area of the sector is around 8.37758 cm.

**Step 2: Find the area of the triangle.**

Using the trigonometric version of the area of a triangle:

$$A = \frac{1}{2} ab \sin C$$

We have established that the radii are equal to the sides of the triangle. We can then put our information in the formula:

$$A = \frac{1}{2}(4)(4)\sin 60°$$

$$A = 8 \sin 60°$$

$$A \approx 6.9282 \text{ cm}$$

**Step 3: Subtract the area of the triangle from the area of the sector.**

When we subtract the area of the triangle from the area of the sector, we get the area of the shaded region.

**Area of the shaded region = Area of the sector - Area of the triangle**

$$A = 8.37758 - 6.9282$$
$$A = 1.44938 \text{ cm}$$

From this we get the area of **one** shaded region. Since there are **six** sectors formed when we divide the regular hexagon into equilateral triangles, we multiply our answer by 6.

**Total Area of the shaded region = (6)(1.44938 cm)**

$A \approx 8.696 \text{ cm} \approx 8.7 \text{ cm}$

**The total area of the shaded region is around 8.7 cm.**

After performing both methods, we get the same answer.



Algebraic Techniques with Trigonometric Functions

In Handout 1.4 page 10, we briefly reviewed the difference between $\sin^2 \theta$ and $2 \sin \theta$. At the end of the section, we will do an example to verify this. This can also be applied to the other trigonometric functions. This will be important since we will now use algebraic techniques with trigonometric expressions. Incorrectly mistaking examples such as the above ($\sin^2 \theta$ and $2 \sin \theta$) can result in wrong calculations.

Let's take a look at an example.

***Example 1.5g***

Perform the indicated operation and simplify.

$$(\cos 2\theta + 2)(\cos 2\theta + 3)$$



Many people dive right into the problem and can make mistakes. For example, when performing the FOIL method, he or she might write $\cos (4\theta)^2$ for the beginning. This is incorrect since it would focus on the argument ($2\theta$) instead of the entire function of $\cos 2\theta$ which would be [($\cos 2\theta$)].

Let's take a quick look at an example where parentheses play a role with trigonometric functions before we do our example:

a) $f(x) = \sin 2x^3 = \sin (2x^3)$

b) $f(x) = \sin (2x)^3 = \sin (8x^3)$

c) $f(x) = (\sin 2)x^3 = (\sin 2)(x^3)$

In our example above that has cos 2θ, we focus on the entire function similar to **a)** in the table instead of the argument (2θ) which is seen in b). So if we were to multiply (sin $2x^3$)(sin $2x^3$), we would get **$\sin^2 (2x^3)$** not **$\sin 4x^6$**.

We multiply the two functions (cos 2θ)(cos 2θ) to get $\cos^2 2\theta$. Therefore, 2θ remains the same and is not multiplied to get $4\theta^2$.

**Step 1: Identify the trigonometric function and rewrite it as a variable.**

Here we have ***cos 2θ***. To simplify, **we rewrite cos 2θ as x**.

Let cos 2θ = x.

**Step 2: Rewrite the equation using the variable.**

$$\mathbf{(x + 2)(x + 3)}$$

Now our equation is simplified and we can multiply the binomials.

**Step 3: Perform the indicated operation.**

When we use the FOIL method, we get **$x^2 + 5x + 6$**.

**Step 4: Substitute our original trigonometric function into the equation.**

Since cos 2θ = x, we replace x with cos 2θ:

$$\cos^2 2\theta + 5\cos 2\theta + 6$$

Here we have our final answer. As you can see, if you separated and focused on the argument (2θ) as we discussed in the beginning, it would have led you to the wrong answer.

***<u>Example 1.5h</u>***

Perform the indicated operation and simplify.

$$\tan 3\theta\,(\tan 4\theta - 2)$$

A common mistake is to multiply tan 3θ and tan 4θ together to write tan 12θ or tan $(12\theta)^2$. In order to avoid that use our substitution technique from our previous example.

Let **tan 3θ** = **x**
Let **tan 4θ** = **y**

$$\tan 3\theta\,(\tan 4\theta - 2)$$

$$x(y-2)$$
$$= xy - 2x$$

After performing our multiplication with the algebraic variables, we put our trigonometric functions in place of the variables to get our final solution.

$$= (\tan 3\theta)(\tan 4\theta) - 2(\tan 3\theta)$$

$$= \mathbf{(\tan 3\theta)(\tan 4\theta) - 2\tan 3\theta}$$

***Example 1.5i***

Now we will do another example using fractions.

$$\frac{4}{\sin 5\theta} + \frac{2}{3\sin 6\theta}$$

**Let sin 5θ = x**
**Let sin 6θ = y**

**Step 1: Perform the algebraic substitution.**

$$\frac{4}{x} + \frac{2}{3y}$$

**Step 2: Find the least common denominator (if the denominators are different; if they are the same you can go ahead and perform the operation).**

Since the denominators are different, we have to find the least common denominator. **The LCD is 3xy**.

To get that multiply the numerator and denominator by: 3y for $\left(\frac{4}{x}\right)$ and x for $\left(\frac{2}{3y}\right)$

We then get

$$\frac{4(3y)}{x(3y)} + \frac{2(x)}{(3y)(x)}$$

$$= \frac{12y}{3xy} + \frac{2x}{3xy}$$

$$= \frac{12y + 2x}{3xy}$$

**Step 3: Substitute our original trigonometric functions back into the equation.**

$$= \frac{12y + 2x}{3xy}$$

$$= \frac{12(sin\ 6\theta) + 2(sin\ 5\theta)}{3[(sin\ 5\theta)(sin\ 6\theta)]}$$

$$= \frac{12\ sin\ 6\theta\ + 2\ sin\ 5\theta}{(3\ sin\ 5\theta)(3\ sin\ 6\theta)}$$

***<u>Example 1.5j</u>***

Factor $\tan^2\theta + 6 \tan \theta + 5$.

**Step 1: Perform the algebraic substitution.**

Let $\tan \theta = x$

We then get:

**$(x^2 + 6x + 5)$**

**Step 2: Factor out the polynomial.**

**$(x^2 + 6x + 5) = (x+5)(x+1)$**

**Step 3: Substitute our original trigonometric function back into the factored polynomial.**

$(\tan\theta + 5)(\tan\theta + 1)$

Please remember the rules and avoid the mistakes we previously discussed:

| | | |
|---|---|---|
| $5(\tan\theta) = 5\tan\theta$ | not | $\tan 5\,\theta$ |
| $(\tan 2\theta)(\tan 2\theta) = \tan^2 2\theta$ | not | $\tan 4\theta^2$ |

With practice, it will be easier to evaluate these operations without doing substitution, but when in doubt, you can always use the substitution method to double check your work.

Let's revisit an example that we started discussing at the beginning of this section to examine the difference between trigonometric functions such as $\cos^2\theta$ vs. $2\cos\theta$.

Let $\theta = 60^\circ$. From our special triangles, we learned that $\mathbf{\cos 60^\circ = \frac{1}{2}}$.

| | | | | | | |
|---|---|---|---|---|---|---|
| $\cos^2\theta$ | = | $(\cos\theta)(\cos\theta)$ | = | $(\cos 60^\circ)(\cos 60^\circ)$ | = | $\mathbf{\frac{1}{4}}$ |
| $2\cos\theta$ | = | $2(\cos\theta)$ | = | $2(\cos 60^\circ)$ | = | **1** |