Topic Review
Precalculus Handout Addendum to 1.2
The Triangle Inequality

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The Triangle Inequality

Let a and b be real numbers,

| a + b | ≤ | a | + | b |

Alternatively the Reverse Triangle Inequality:

| a - b | ≥ | a | - | b |

Continuing our discussion on absolute value we will cover an important concept in mathematics called The Triangle Inequality.

The Triangle Inequality states that the length of the two sides | a | + | b | is greater than or equal to the length of the third side | a + b |.
(The absolute value of the sum of | a + b| is always less than or equal to the absolute value of each individual sum | a | + | b |).

However, note that | a + b | ≠ | a | + | b | since having opposite signs for real numbers **a and b** will result in different values for each side.

For example, let a = -1 and b = 2

| -1 + 2 | ≠ | -1 | + | 2 |

1 ≠ 3

The left side will result in subtraction while the numbers on the right side are added.

When the values have the **same sign** (both positive or both negative),

$|a + b| = |a| + |b|$

For example, let $a = 1$ and $b = 2$

$|1 + 2| = |1| + |2|$

Both sides will equal 3.

For our next example we'll use negative values, let $a = -1$ and $b = -2$

$|-1 + (-2)| = |-1| + |-2|$

$|-3| = 1 + 2$

Again, both sides will equal 3.

Since the statement $|a + b| = |a| + |b|$ is only true when the two values have the same sign, but not when they have opposite signs, the statement can be true only if:

$|a + b| \leq |a| + |b|$

Remember from the Absolute Value worksheet that

$$-|a| \le a \le |a|$$

is true because if we substitute a real number for “a”, the statement is true.

The statement is also true for “b”:

$$-|b| \le b \le |b|$$

If k > 0,

$$|a| = k \quad \text{if and only if} \quad a = \pm k$$

If $k = \pm 1$, and we put the values into $|a|$, then $|-1|$ and the $|1|$ both equal 1 (k).

So if we add the two statements above for a and b:

$$-|a| \le a \le |a|$$
$$+$$
$$-|b| \le b \le |b|$$

$$-[\,|a + b|\,] \le a + b \le |a| + |b|$$

Also note the absolute value property:

$|a| \le k$ if and only if $\quad -k \le a \le k$

From adding the two statements for a and b, we can take the result and apply it to the absolute value property $|a| \le k$:

$$-[\,|a + b|\,] \quad \le \quad a + b \quad \le \quad |a| + |b|$$

$$-k \quad \le \quad a \quad \le \quad k$$

where **a** would be substituted for **a + b** and **k** would be substituted for **| a | + | b |**. From this we get the Triangle Inequality:

$$|a + b| \quad \le \quad |a| + |b|$$

$$|a| \quad \le \quad k$$

Now let’s apply the Triangle Inequality in an example.

Example 1: If | x - 5 | < 0.3 and | y - 7 | < 0.6, estimate | (x + y) - 12 | using the Triangle Inequality.

With the Triangle Inequality we note:

$a = x - 5$

$b = y - 7$

$| a + b | \leq | a | + | b |$

$| (x + y) - 12 | = | (x - 5) + (y - 7) |$

$| (x + y) - 12 | \leq | x - 5 | + | y - 7 |$

$| (x + y) - 12 | < 0.3 + 0.6$

$| (x + y) - 12 | < 0.9$