Calculus 2 Review
Handout 1.1
Review of Indefinite Integration and Basic Integration Rules

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As we dive into Calculus II, we will begin with a review of indefinite and definite integration. These topics will refresh what was learned at the end of Calculus I and allow us to smoothly transition to applications of integral calculus (where Calculus II usually begins). Please note that this is a review and more thorough definitions are covered in the Calculus I notes.

Indefinite Integration (Antidifferentiation)

Let's begin with a review of a few concepts regarding indefinite integration (or antidifferentiation.

Suppose we have the function **F(x) =** $\boldsymbol{x^5}$. Through the Power Rule in differentiation, we find the function's derivative: $\boldsymbol{f}$**(x) = 5x⁴**.

The function F(x) is **an** antiderivative of f(x) on an interval. F(x) is an antiderivative of f(x) if and only if the derivative of F(x) equals f(x). We can describe this as

$$\boldsymbol{F(x) = \int f(x)\,dx}$$

if and only if

$$\boldsymbol{F'(x) = f(x)}$$

Note that "**an**" is highlighted above. This is because a function can have many antiderivatives. For our function f(x) = $5x^4$, it can have antiderivatives such as

$\boldsymbol{x^5}$
$\boldsymbol{x^5 - 0.01}$
$\boldsymbol{x^5 + e}$

$$\boldsymbol{x^5 + 10}$$
$$\boldsymbol{x^5 - 473}$$

etc.

As a result, we note that the function **G(x) = $\boldsymbol{x^5}$ + C** is the General Antiderivative of the function $f(x) = 5x^4$. C represents the **constant of integration**. The function $G(x) = x^5 + C$ represents the family of antiderivatives for our function $f(x) = 5x^4$. We can observe several members of this family highlighted above where C can represent -0.01, e, 10 or -473.

With this information, we can extend the definition of the indefinite integral as

$$\int \boldsymbol{f(x)\,dx = F(x) + C}$$

We use our example of $f(x) = 5x^4$ and apply it to this definition:

$$\int \boldsymbol{5x^4\,dx = x^5 + C}$$

Differentiation and integration are inverse processes of each other. You can differentiate your answer to check your solution.

Let's take a look at an example.

Example 1.1a

Evaluate the antiderivative (indefinite integral) and verify your solution through differentiation.

$$\int \boldsymbol{(x^2 + 4x + 2)\,dx}$$

This is a polynomial function and we just need to use our integration rules to integrate it.

Step 1: Rewrite our original integral.

$$\int \boldsymbol{(x^2 + 4x + 2)\,dx} = \int \boldsymbol{x^2\,dx} + \int \boldsymbol{4x\,dx} + \int \boldsymbol{2\,dx}$$

Step 2: Apply the Power Rule for Integration and the Constant Multiple Rule.

Power Rule:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1$$

Constant Multiple Rule:

$$\int kf(x)\,dx = k\int f(x)\,dx$$

$= \int x^2\,dx + \int 4x\,dx + \int 2\,dx$

$= \int x^2\,dx + 4\int x\,dx + \int 2\,dx$

$= \frac{x^3}{3} + \frac{4x^2}{2} + 2x + C$

Step 3: Simplify.

$= \frac{1}{3}x^3 + 2x^2 + 2x + C$

Step 4: Differentiate our result for verification.

Now we can differentiate to verify our result.

$$\frac{d}{dx}\left[\frac{1}{3}x^3 + 2x^2 + 2x + C\right]$$

$= \frac{1}{3}\frac{d}{dx}[x^3] + 2\frac{d}{dx}[x^2] + 2\frac{d}{dx}[x] + \frac{d}{dx}[C]$

$= \frac{1}{3}(3x^2) + 2(2x) + 2(1) + 0$

$= x^2 + 4x + 2$

Now let's try an example involving trigonometric functions.

Example 1.1b

Evaluate the antiderivative (indefinite integral) and verify your solution through differentiation.

$$\int \frac{\cos x}{\sin^2 x} dx$$

Step 1: Rewrite the integrand.

In this case, we can write them as products.

$$\int \left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right) dx$$

Step 2: Simplify further using trigonometric identities.

From our trigonometric identities, we learned that

$$\frac{1}{\sin x} = \csc x$$

$$\frac{\cos x}{\sin x} = \cot x$$

So we get:

$$\int \csc x \cot x \; dx$$

Step 3: Integrate.

From our basic integration rules, we learned that

$$\int \csc x \cot x \; dx = -\csc x + C$$

Step 4: Differentiate our result for verification.

$$\frac{d}{dx}[-\csc x + C]$$

= (-1)$\frac{d}{dx}$ [csc x] + $\frac{d}{dx}$ [C]

= (-1)(-csc x cot x) + 0

= csc x cot x

From here, we have the simplified version of the integrand. We can expand it further to get the original integral in the problem.

Brief Overview of Differential Equations

We're going to briefly touch on differential equations since we will be covering them in the next section when solving initial value problems. Courses later on will go more in depth on ordinary and partial differential equations.

A **differential equation** is an equation involving derivatives or differentials. A few examples would be

$$\frac{dy}{dx} = 4x + 1$$

$$\frac{dy}{dt} = 6$$

$$y' = \cos x$$

We learned in differentiation that

$$\boldsymbol{\frac{dy}{dx} = f'(x)}$$

denotes the derivative of the function y with respect to x.

The differential equation can also be written in **differential** form where

$$\boldsymbol{dy = f'(x)dx}$$

As you already noticed, the differential form is what we use throughout integration. To get $f'(x)$, just find the derivative of the function.

For example, if we have the function $\boldsymbol{y = 7x^2 + 9x}$ we can find the differential **d*y***:

$$\boldsymbol{dy = f'(x)dx}$$

$$y = 7x^2 + 9x$$

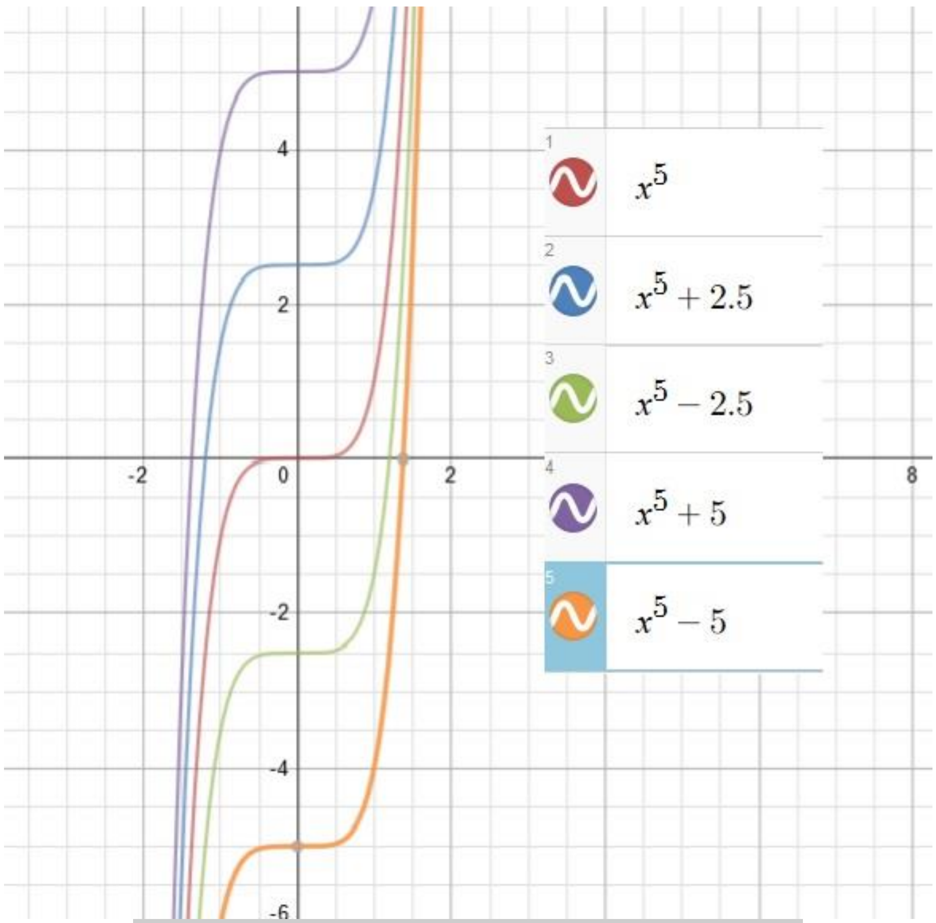
$$f'(x) = 14x + 9$$

$$\mathbf{d}\boldsymbol{y}\mathbf{ = (14x + 9)dx}$$

Particular Solutions

We previously learned that there are many solutions to $y = \int f(x)\,dx$ with each solution having different constants. Since it is the constants that differ for the antiderivatives of f(x), we can observe that each antiderivative is a vertical translation of each other.

For example, in our example above we can graph a few of the antiderivatives for $y = \int 5x^4\,dx$ and observe a few of its solutions. A few values of C included here are 2.5, -2.5, 5, and -5.



F(x) = x^5 + C
(This is the general solution)

The **general solution** shows us the antiderivatives for $5x^4$ and how these antiderivatives and their different integer values for C can be a solution to the given differential equation, $\frac{dy}{dx} = 5x^4$.

We will now learn how to find a particular solution. In order to find a particular solution, we will need to know the **initial condition**.

The initial condition provides the information for one specific value of x for y = F(x). In our above graph, we can observe several values for C. With the initial condition, we can find which value of C inputted into the particular solution passes through the point (x, y).

With the information from the general solution and the initial condition, we can find the particular solution. This type of problem is called an **initial value problem**. Let's take a look at an example so that we can visualize this.

Example 1.1c

Find the general solution of F'(x) = 2x + 3 and determine the particular solution that satisfies the initial condition F(3) = 9.

Step 1: Find the general solution.

F'(x) = 2x + 3

We begin with F'(x) which we previously learned is the derivative of F(x). In order to find the general solution, we have to integrate F'(x) to get F(x).

F(x) = $\int (2x + 3)\,dx$
F(x) = $2\int x\,dx + \int 3\,dx$
F(x) = $2\left(\frac{x^2}{2}\right) + 3x + C$

F(x) = x^2 + 3x + C
This is our general solution.

Step 2: Use the initial condition to solve for C.

We are given the initial condition F(3) = 9. We saw that a function can have many antiderivatives. However, we now know that the curve that we are looking for passes through the point (3, 9). The initial condition allows us to pinpoint and find the particular solution.

We use the information from our initial condition and apply it to our general solution to solve for C.

F(x) = x^2 + 3x + C
F(3) = $(3)^2$ + 3(3) + C

9 = 9 + 9 + C
9 = 18 + C
C = -9

Step 3: Determine the particular solution.

Now that we've obtained C, we can determine the particular solution that satisfies the initial condition.

The particular solution that satisfies the initial condition F(3) = 9 is **F(x) = $x^2 + 3x - 9$**.

Example 1.1d

Solve the following initial value problems.

a) $f'(x) = 5x^2, f(3) = 20$
b) $y'' = 8x - 2\ ;\ y'(0) = 1; y(1) = 4$

Similar to above, we have to find the general solution, utilize our initial condition, and find the particular solution that satisfies the initial condition.

For a) $f'(x) = 5x^2, f(3) = 20$

Step 1: Integrate the derivative to get the general solution.

$F(x) = \int 5x^2\ dx$
$F(x) = 5\int x^2\ dx$
$F(x) = 5\left(\frac{x^3}{3}\right) + C$

$$\mathbf{F(x) = \frac{5}{3}x^3 + C}$$

Step 2: Use the initial condition to solve for C.

$f(3) = 20$

Similar to our previous example, we use the information from the initial condition to solve for C.

$$\mathbf{F(x) = \frac{5}{3}x^3 + C}$$

$$\mathbf{F(3) = \frac{5}{3}(3)^3 + C}$$

$$\mathbf{F(3) = \frac{5}{3}(27) + C}$$

$$\mathbf{F(3) = \frac{135}{3} + C}$$

$$\mathbf{20 = 45 + C}$$

C = -25

Step 3: Determine the particular solution.

The particular solution that satisfies the initial condition F(3) = 20 is **F(x) =** $\boldsymbol{\frac{5}{3}x^3 - 25}$**.** If you want to factor it out you can write it as **F(x) =** $\frac{5}{3}$ **(x^3 - 15)**.

For b) $\boldsymbol{y'' = 8x - 2; y'(0) = 1; y(3) = 10}$

When we solve this initial value problem, we have to perform an additional step since we are given the second derivative, y'' (unlike in our previous example where we began with y').

Step 1: Integrate the second derivative to get $\boldsymbol{y'}$**.**

$$y' = \int (8x - 2)\,dx$$

$$y' = 4x^2 - 2x + C$$

Step 2: Use the initial condition $\boldsymbol{y'(0) = 1}$ **and solve for C.**

$$y' = 4x^2 - 2x + C$$

$$1 = 4(0)^2 - 2(0) + C$$

C = 1

We can now plug in C to get $\boldsymbol{y' = 4x^2 - 2x + 1}$

Step 3: Perform the same steps to get the general solution.

Now that we have y', we can integrate it to get y.

$$y = \int (4x^2 - 2x + 1)\,dx$$

$$y = \frac{4}{3}x^3 - x^2 + x + C$$

Use the initial condition $\boldsymbol{y(3) = 10}$ **and solve for C.**

$10 = \frac{4}{3}(3)^3 - (3)^2 + 3 + C$

$10 = 30 + C$

C = -20

The particular solution is $\boldsymbol{y = \frac{4}{3}x^3 - x^2 + x - 20}$**.**