Calculus 2
Handout 1.1B (to accompany Handout 1.1)
Vertical and Additional Linear Motion Problems

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We first encountered vertical motion problems during the chapter on differentiation when we discussed rates of change. We usually start with the position function which gives us the position of the object relative to the origin.

For example, if a person is standing on a bridge with a river running under it and drops a pebble from a height of 25 meters, the position function

$$\boldsymbol{s(t) = -4.9t^2 + 25}$$

represents the height $\boldsymbol{s}$ (relative to the origin) as a function of time $\boldsymbol{t}$. In our example the height, s, is measured in meters and the time, t, is measured in seconds.

To get the velocity function, we found the derivative of the position function. In our example, the velocity function would be $s'(t) = v(t) = -9.8t$.

To get the acceleration function, we found the derivative of the velocity function or the second derivative of the position function. In our example, the acceleration function would be

$$s''(t) = v'(t) = a(t) = -9.8$$

We are now going to revisit this topic and apply integration methods to linear motion problems. In this case we are given the acceleration due to gravity and we will need to find velocity and position functions.

Let's take a look at an example.

<u>Example 1</u>

A model rocket was launched upward from an initial height of 112 feet with an initial velocity of 96 feet per second. The acceleration due to gravity is given by a(t) = -32 feet per second per second. Neglect air resistance.

a) Find the position function s(t) expressing height *s* as a function of time *t*.
b) How long will it take the rocket to reach the ground?

Let's begin with part a and find the position function.

<u>Part A</u>

Step 1: Organize the known information.

We know that a(t) = -32 ft/s^2.

The initial height, s_0 is 112 feet.

The initial velocity, v_0 is 96 feet per second.

Step 2: Integrate the acceleration function to get the velocity function.

We learned in differentiation that acceleration is $s''(t)$ or $v'(t)$. By integrating $v'(t)$, we can get $v(t) = s'(t)$.

From our information, $s''(t) = a(t) = -32\ ft/s^2$.

To get the velocity function, we integrate $s''(t)$.

$$s'(t) = \int -32\ dt$$

$$s'(t) = -32t + C_1$$

The initial velocity at t = 0 is 96 ft/s. In other words, $s'(0) = 96$.

We put this into the equation to find C_1.

$$s'(0) = 96$$

$$96 = -32(0) + C_1$$

$$C_1 = 96$$

Step 3: Integrate the velocity function to find the position function.

In our previous step we found the velocity function, $s'(t) = -32t + 96$. We now integrate the velocity function to get the position function s(t).

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$$s(t) = \int (-32t + 96)\, dt$$

$$s(t) = -16t^2 + 96t + C_2$$

Similar to the previous step, we find C_2 at t = 0. The initial height where the rocket was launched is 112 feet.

$$s(0) = 112$$

$$112 = -16(0)^2 + 96(0) + C_2$$

$$C_2 = 112$$

We put C_2 into s(t) to get the position function.

$$\boldsymbol{s(t) = -16t^2 + 96t + 112}$$

Now that we found the position function we can now calculate when the rocket hits the ground.

<u>Part B</u>

To find the time when the rocket hits the ground, let s(t) = 0 and solve for t.

$$\boldsymbol{-16t^2 + 96t + 112 = 0}$$

$$-16(t^2 - 6t - 7) = 0$$

$$-16(t + 1)(t - 7) = 0$$

$$t = -1, 7$$

Since time is positive (t ≥ 0) in this case, the rocket hits the ground 7 seconds after it is launched.

<u>Example 2</u>

A ball is thrown vertically upward from a height of 4 meters with an initial velocity of 12 meters per second. What is the maximum height of the ball?

The acceleration due to gravity is -9.8 m/s^2. Neglect air resistance.

Remember that an object reaches maximum height when the velocity is <u>zero</u>.

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Step 1: Integrate the acceleration function to get the velocity function.

$$s'(t) = \int -9.8\, dt$$

$$s'(t) = -9.8t + C_1$$

The initial velocity at t = 0 is 12 m/s. In other words, $s'(0) = 12$.

We put this into the equation to find C_1.

$$s'(0) = 12$$

$$12 = -32(0) + C_1$$

$$C_1 = 12$$

$$\boldsymbol{s'(t) = -9.8t + 12}$$

Step 2: Find the time when the ball reaches maximum height.

In the beginning of the example we learned that an object reaches maximum height when the velocity is zero.

Let $s'(t) = 0$ and solve for t.

$$-9.8t + 12 = 0$$

$$-9.8t = -12$$

$$t = \frac{-12}{-9.8}$$

$$\boldsymbol{t \approx 1.22 \text{ seconds}}$$

We now know that at $t \approx 1.22$ seconds, the ball reaches maximum height.

Step 3: Find the position function to determine the ball's height at $t \approx 1.22$ seconds.

First we have to find the position function. Similar to the last example, we integrate the velocity function to determine the position function.

$$s(t) = \int (-9.8t + 12)\, dt$$

$$s(t) = -4.9t^2 + 12t + C_2$$

Similar to above, we find C_2 at t = 0. The initial height where the ball was thrown is 4 meters.

$$s(0) = 4$$

$$4 = -4.9(0)^2 + 12(0) + C_2$$

$$C_2 = 4$$

$$s(t) = -4.9t^2 + 12t + 4$$

At 1.22 seconds, the ball reached maximum height. We just need to determine the position of the ball at t ≈ 1.22 seconds.

$$s(t) = -4.9t^2 + 12t + 4$$

$$s(1.22) = -4.9(1.22)^2 + 12(1.22) + 4$$

$$s(1.22) \approx 11.35\ meters$$

The ball's maximum height is 11.35 meters.

We can also apply these analytical methods to additional linear motion problems (such as rectilinear motion, motion along a straight line).

<u>Example 3</u>

A customer at a dealership is examining a car to test drive. The salesperson approaches the customer and explains the car's performance where it will take 12 seconds to accelerate from 20 kilometers per hour to 80 kilometers per hour. For this problem, assume that the car moves with constant acceleration.

a) What is the acceleration in meters per second per second?
b) What is the distance the car travels during the 12 seconds it accelerates?

Let's begin with a.

<u>Part A</u>

Step 1: Find the velocity and distance functions with respect to time.

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Since the car moves with constant acceleration, let $\boldsymbol{a(t) = c}$. (It'll be easier since the c can represent **c**onstant acceleration). We integrate this to get our velocity function.

<u>Velocity function</u>

$$v(t) = \int a(t)dt = \int c\, dt$$

$$\boldsymbol{v(t) = ct + v_0}$$

<u>Distance function</u>

$$x(t) = \int v(t)dt = \int (ct + v_0)\, dt$$

$$\boldsymbol{x(t) = \frac{1}{2}ct^2 + v_0 t + x_0}$$

Note that in the velocity function, the initial velocity, v_0, represents C, the constant of integration. In our problem, v_0 is specified.

In the distance function, x_0, is the constant of integration, C.

These two (v_0 and x_0) will be explained in Step 3 and Part B.

Now we have the following information:

Acceleration function: $\boldsymbol{a(t) = c}$
Velocity function: $\boldsymbol{v(t) = ct + v_0}$
Distance function: $\boldsymbol{x(t) = \frac{1}{2}ct^2 + v_0 t + x_0}$

Step 2: Perform the unit conversions to match the units in the solution.

Since our units are in kilometers per hour and the units in the solution are in meters per second per second, we have to convert kilometers to meters and one hour to seconds.

Here are the conversion units:
1 kilometer = 1000 meters
1 hour = 60 minutes = 3600 seconds

From our information:

20 kilometers = 20 x 1000 meters = 20000 meters
80 kilometers = 80 x 1000 meters = 80000 meters

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We already know that 1 hour is 3600 seconds. We just simplify our units.

$$\frac{20000\ meters}{3600\ seconds} = \frac{200}{36}\ m/s$$

$$\frac{80000\ meters}{3600\ seconds} = \frac{800}{36}\ m/s$$

Now we have the information to find acceleration.

Step 3: Use the velocity function to find acceleration.

The final velocity, v, is $\frac{800}{36}\ m/s$.

The initial velocity, v_0, is $\frac{200}{36}\ m/s$.

The time, t, is 12 seconds.

<u>The velocity function</u>

$$\boldsymbol{v(t) = ct + v_0}$$

We input our information into the velocity function:

$$\frac{800}{36} = \boldsymbol{c(12)} + \frac{200}{36}$$

$$\frac{600}{36} = \boldsymbol{12c}$$

$$c = \frac{600}{36} \cdot \frac{1}{12}$$

$$c = \frac{50}{36} \approx 1.39\ m/s^2$$

The car's acceleration is 1.39 m/s².

Now we can move to b.

<u>Part B</u>

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Since we found the majority of the information in part a, we just need to calculate the distance the car travels in those 12 seconds.

Using the information gathered and the distance function, input and calculate the distance traveled.

$$\boldsymbol{x(t) = \frac{1}{2}ct^2 + v_0t + x_0}$$

We now find $x(12)$.

Our information gathered:

The initial displacement x_0 is zero. We are finding the distance from the starting point x_0 to its final position after 12 seconds which is the value of $x(12)$.

The final velocity, v, is $\frac{800}{36}\ m/s$.

The initial velocity, v_0, is $\frac{200}{36}\ m/s$.

The time, t, is 12 seconds.

The acceleration, c, is 1.39 m/sec². (If you want to be more precise, you can put c = $\frac{50}{36}$.

$$\boldsymbol{x(12) = \frac{1}{2}ct^2 + v_0t + x_0}$$

$$\boldsymbol{x(12) = \frac{1}{2}(\frac{50}{36})(12)^2 + \frac{200}{36}(12)}$$

$$\boldsymbol{x(12) = \frac{3600}{36} + \frac{2400}{36}}$$

$$\boldsymbol{x(12) = \frac{6000}{36} \approx 167\ m}$$

The distance traveled during the 12 seconds is 167 meters.

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