



Calculus II Review Handout 1.2 Sigma Notation and Area

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We will continue our review of integral calculus. In this section we will cover summation notation and finding the area of a region.

Sigma Notation

We will begin by briefly discussing **sequences**, **series**, and **sigma notation**. If you recall from earlier mathematics courses, a **sequence** is an ordered set of objects (in our case, the objects will be numbers) containing terms which follow a certain pattern. In a series, each term in the sequence is summed together (we will cover this in greater detail when we look at **infinite series**).

Sigma notation is denoted by

$$\sum_{i=1}^n a_i = a_1 + a_2 + a_3 \dots + a_n$$

where

i is the index of summation (different textbooks may use other letters for the index of summation, but i , j , and k are the ones commonly used)

a_i is the i th term of the sum

n is the upper bound (limit) of summation

1 is the lower bound (limit) of summation (this is the case for the above notation; however, the lower bound of summation can be any integer less than or equal to the upper bound)

Regarding the final point above discussing the lower bound of summation, let's say $n = 3$, i can be any integer equal to or less than 3.

Let's take a look at a few examples:

1. For this example, we just start with the lower bound and sum up the numbers until the upper bound. Since $a_i = i$ in our example, we just need to sum up the numbers starting with 1 and finishing at 3.

$$\begin{aligned}\sum_{i=1}^3 i &= 1 + 2 + 3 \\ &= 6\end{aligned}$$

Later on in this handout we will learn a summation formula that we can apply to this series:

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^3 i = \frac{3(3+1)}{2}$$

$$\begin{aligned}\sum_{i=1}^3 i &= \frac{12}{2} \\ &= 6\end{aligned}$$

Now let's take a look at an example of a series that is more involved:

$$\sum_{k=1}^6 k^2 + 1$$

2. For this example, we note that the lower bound is 1 and the upper bound is 6. However, unlike the previous example, we now have $k^2 + 1$. Our a_i is $k^2 + 1$.

When we do the summation we will to put each number from the lower bound to the upper bound (basically from 1 to 6) into $k^2 + 1$ and then we can sum everything up.

$$\sum_{k=1}^6 k^2 + 1$$

$$= (1^2 + 1) + (2^2 + 1) + (3^2 + 1) + (4^2 + 1) + (5^2 + 1) + (6^2 + 1)$$

$$= 2 + 5 + 10 + 17 + 26 + 37$$

$$= 97$$

We can also use another method for example 2 which we will discuss as we take a look at the properties of summation.

Properties of Summation:

<p>Let k be a constant:</p> $\sum_{i=1}^n k a_i = k \sum_{i=1}^n a_i$
$\sum_{i=1}^n (a_i \pm b_i) = \sum_{i=1}^n a_i \pm \sum_{i=1}^n b_i$

We can use the second property for example 2 above:

$$\sum_{k=1}^6 k^2 + 1 = \sum_{k=1}^6 k^2 + \sum_{k=1}^6 1$$

Now let's break it down into parts

$$\sum_{k=1}^6 k^2$$

$$= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2$$

$$= 91$$

For this part, we will use an important summation formula which we will see again when we look at the table of summation formulas later on in the handout:

$$\sum_{i=1}^n 1 = n$$

$$\sum_{k=1}^6 1$$

$$= 6$$

When we put it together and apply our property:

$$\sum_{k=1}^6 k^2 + \sum_{k=1}^6 1$$

$$= 91 + 6$$

$$= 97$$

As you can see we get the same result!

Now that we've had some practice, it should be easier for you to look at a sum of terms and present it in sigma notation.

Suppose you are given a sum and need to express it in sigma notation.

For example:

$$\left[4 \left(\frac{1}{5} \right) - 2 \right] + \left[4 \left(\frac{2}{5} \right) - 2 \right] + \left[4 \left(\frac{3}{5} \right) - 2 \right] + \cdots + \left[4 \left(\frac{8}{5} \right) - 2 \right]$$

Step 1: Look for the pattern among the terms to find an expression.

When we look at the terms in the series, everything is the same with the exception of where I placed x :

$$\left[4 \left(\frac{x}{5} \right) - 2 \right]$$

Step 2: Find the values that change to determine the upper and lower bounds of summation.

In our example, we see that the numerator changes (where x is placed). The lowest number is 1 and the largest number is 8.

The lower bound (limit) of summation is **1**.

The upper bound (limit) of summation is **8**.

Step 3: Write the expression of the sum in sigma notation.

Using our information from Steps 1 and 2:

$$\sum_{i=1}^8 4\left(\frac{i}{5}\right) - 2$$

Important Summation Formulas:

$\sum_{i=1}^n 1 = n$	
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$(1) \sum_{i=1}^n c = cn$	$(2) \sum_{i=1}^n i = \frac{n(n+1)}{2}$
$(3) \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$	$(4) \sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$

Let's apply these formulas through examples:

-1-

$\sum_{i=1}^n c = cn$	
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$$= 4(14)$$

$$= 56$$

-2-

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{12} 3i$$

For this example, we will apply the first property of summation we learned above together with this summation formula.

$$\sum_{i=1}^{12} 3i = 3 \sum_{i=1}^{12} i$$

Note that:

$$\sum_{i=1}^{12} i = \frac{12(12+1)}{2} = \frac{156}{2} = 78$$

Put everything together:

$$= (3)(78)$$

$$= \mathbf{234}$$

-3-

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{10} (i+1)^2$$

This example will be more involved so I will break this down step by step.

First, we have to expand the binomial:

$$(i+1)^2 = i^2 + 2i + 1$$

we then write it out as

$$\sum_{i=1}^{10} (i + 1)^2 = \sum_{i=1}^{10} i^2 + 2i + 1$$

Applying all of the formulas and properties that we learned so far:

$$\begin{aligned} & \sum_{i=1}^{10} i^2 + 2i + 1 \\ &= \sum_{i=1}^{10} i^2 + 2 \sum_{i=1}^{10} i + \sum_{i=1}^{10} 1 \end{aligned}$$

Let's evaluate each term separately and then add them up at the end:

$$\begin{aligned} \sum_{i=1}^n i^2 &= \frac{n(n+1)(2n+1)}{6} \\ \sum_{i=1}^{10} i^2 &= \frac{10(10+1)((2 \cdot 10) + 1)}{6} \\ &= \frac{(110)(21)}{6} = \frac{2310}{6} \\ &= \mathbf{385} \end{aligned}$$

Now to the second part:

$$\begin{aligned} 2 \sum_{i=1}^n i &= (2) \left(\frac{n(n+1)}{2} \right) \\ 2 \sum_{i=1}^{10} i &= (2) \left(\frac{10(10+1)}{2} \right) \end{aligned}$$

$$\begin{aligned}
 &= (2) \left[\frac{10(10 + 1)}{2} \right] \\
 &= (2) \left[\frac{110}{2} \right] \\
 &= \mathbf{110}
 \end{aligned}$$

Finally for the last part:

$$\begin{aligned}
 \sum_{i=1}^n 1 &= n \\
 \sum_{i=1}^{10} 1 &= \mathbf{10}
 \end{aligned}$$

We sum all of the terms together:

$$\begin{aligned}
 \sum_{i=1}^{10} i^2 + 2i + 1 \\
 = 385 + 110 + 10 \\
 = \mathbf{505}
 \end{aligned}$$

-4-

Let's do an example with subtraction since we covered addition and the distributive property of multiplication in the previous one.

$$\sum_{i=1}^8 (i^3 - 16)$$

This one should be easy to do since we already had a lot of practice from our previous examples!

$$\sum_{i=1}^n i^3 = \frac{n^2(n + 1)^2}{4}$$

We use our properties:

$$\sum_{i=1}^8 i^3 - \sum_{i=1}^8 16$$

For the first term:

$$\begin{aligned} \sum_{i=1}^8 i^3 &= \frac{n^2(n+1)^2}{4} \\ &= \frac{(8)^2(8+1)^2}{4} \\ &= \frac{(64)(81)}{4} = \frac{5184}{4} \\ &= \mathbf{1296} \end{aligned}$$

For the second term:

$$\sum_{i=1}^n c = cn$$

$$\begin{aligned} \sum_{i=1}^8 16 &= (16)(8) \\ &= \mathbf{128} \end{aligned}$$

Perform the summation:

$$\begin{aligned} \sum_{i=1}^8 i^3 - \sum_{i=1}^8 16 &= 1296 - 128 \\ &= \mathbf{1168} \end{aligned}$$

To finish up this section on sigma notation, we will simplify the process of evaluating a sum for several values of n .

So far we only had to solve for one value of n in which we applied our summation formulas and properties. Suppose you are given a sum to evaluate for different values of n . For example,

$$\sum_{i=1}^n \frac{(i-1)}{4n^2}$$

and I ask you to evaluate the sum for **$n = 2, 4, 6, 8,$ and 10** . You can do what we did earlier and evaluate the sum for each individual n separately, but it may become tedious. Instead, we will now learn a way to rewrite the expression and substitute the given values into n .

Let's take a look at an example.

Evaluate the sum

$$\sum_{i=1}^n 4n(i^2 + 3)$$

for $n = 2, 4, 6,$ and 8 .

Step 1: Apply the summation formulas and properties and rewrite the expression.

First we distribute the $4n$:

$$\sum_{i=1}^n k a_i = k \sum_{i=1}^n a_i$$

$$4n \sum_{i=1}^n (i^2 + 3)$$

the constant k is $4n$ in our example.

Next we use the second property of summation that we learned

$$\sum_{i=1}^n (a_i \pm b_i) = \sum_{i=1}^n a_i \pm \sum_{i=1}^n b_i$$

and we get

$$4n \left[\sum_{i=1}^n i^2 + \sum_{i=1}^n 3 \right]$$

Step 2: Simplify the expression further using the summation formulas.

Now we will rewrite the expressions inside the brackets.

First we begin with

$$\begin{aligned} \sum_{i=1}^n i^2 \\ = \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

Next we rewrite

$$\begin{aligned} \sum_{i=1}^n 3 \\ = 3n \end{aligned}$$

Step 3: Simplify to get the final rewritten expression.

Now that we simplified each part, we can now put everything together:

$$\begin{aligned} 4n \left[\left(\frac{n(n+1)(2n+1)}{6} \right) + 3n \right] \\ = 4n \left[\left(\frac{(n^2+n)(2n+1)}{6} \right) + 3n \right] \\ = 4n \left[\left(\frac{2n^3+3n^2+n}{6} \right) + 3n \right] \end{aligned}$$

$$\begin{aligned}
 &= 4n \left(\frac{2n^3 + 3n^2 + n}{6} \right) + 12n^2 \\
 &= \left(\frac{2n(2n^3 + 3n^2 + n)}{3} \right) + 12n^2 \\
 &= \frac{4n^4 + 6n^3 + 2n^2}{3} + 12n^2
 \end{aligned}$$

You can also further simplify it to:

$$\begin{aligned}
 &= \frac{4n^4 + 6n^3 + 2n^2}{3} + \frac{36n^2}{3} \\
 &= \frac{4n^4 + 6n^3 + 38n^2}{3}
 \end{aligned}$$

Sometimes students make mistakes when finding common denominators so if you want to keep it simple, you can leave it as is and solve. Either way you should get the same answer.

Now that we have a more simplified expression, we can just plug in our values for n and evaluate the sum.

For $n = 2$:

$$= \frac{4(2)^4 + 6(2)^3 + 2(2)^2}{3} + 12(2)^2$$

$$= \frac{64 + 48 + 8}{3} + 48$$

$$= \frac{120}{3} + 48$$

$$= 40 + 48$$

$$= \mathbf{88}$$

If you further simplified using common denominators you can also use:

$$= \frac{4(2)^4 + 6(2)^3 + 38(2)^2}{3}$$

$$= \frac{64 + 48 + 152}{3}$$

$$= \frac{264}{3}$$

$$= \mathbf{88}$$

Let's evaluate the sum using the summation formulas that we used earlier in this handout (here we let $n = 2$):

$$\sum_{i=1}^n 4n(i^2 + 3)$$

$$= \sum_{i=1}^2 8(i^2 + 3)$$

$$= 8 \sum_{i=1}^2 (i^2 + 3)$$

$$= 8 \left[\sum_{i=1}^2 i^2 + \sum_{i=1}^2 3 \right]$$

Let's evaluate the sums inside the bracket:

For the first one:

$$\sum_{i=1}^2 i^2$$

$$= \frac{2(2+1)(2(2)+1)}{6}$$

$$= \frac{2(3)(5)}{6}$$

$$= \frac{30}{6}$$

$$= \mathbf{5}$$

For the second one:

$$\sum_{i=1}^2 3$$

$$= 3(2)$$

$$= \mathbf{6}$$

Now we go back to our sums:

$$= 8 \left[\sum_{i=1}^2 i^2 + \sum_{i=1}^2 3 \right]$$

$$= 8 \left(\sum_{i=1}^2 i^2 \right) + 8 \left(\sum_{i=1}^2 3 \right)$$

$$= 8(5) + 8(6)$$

$$= 40 + 48$$

$$= \mathbf{88}$$

We get the same answer! The only difference is that having the simplified expression with only the ***n*** allows us to save time and plug in the values for ***n*** instead of going through this longer process. However, it's your choice to approach the problem with whatever is most comfortable for you. As long as you follow one of the paths above to get to the correct conclusion, then everything should be fine.

Now let's finish up the problem.

For *n* = 4:

$$= \frac{4(4)^4 + 6(4)^3 + 2(4)^2}{3} + 12(4)^2$$

$$= \frac{1024 + 384 + 32}{3} + 192$$

$$= \frac{1440}{3} + 192$$

$$= 480 + 192$$

$$= \mathbf{672}$$

Similar to our $n = 2$ example, you can also use:

$$\sum_{i=1}^n 4n(i^2 + 3)$$

$$= \sum_{i=1}^4 16(i^2 + 3)$$

$$16 \sum_{i=1}^4 (i^2 + 3)$$

$$= 16 \left[\sum_{i=1}^4 i^2 + \sum_{i=1}^4 3 \right]$$

$$= 16 \left(\sum_{i=1}^4 i^2 \right) + 16 \left(\sum_{i=1}^4 3 \right)$$

$$= 16(30) + 16(12)$$

$$= 480 + 192$$

$$= \mathbf{672}$$

For $n = 6$:

$$= \frac{4(6)^4 + 6(6)^3 + 2(6)^2}{3} + 12(6)^2$$

$$= \frac{5184 + 1296 + 72}{3} + 432$$

$$= \frac{6552}{3} + 432$$

$$= 2184 + 432$$

$$= \mathbf{2616}$$

Again, to evaluate this you can also use:

$$\sum_{i=1}^n 4n(i^2 + 3)$$

$$= \sum_{i=1}^6 24(i^2 + 3)$$

For $n = 8$:

$$\frac{4(8)^4 + 6(8)^3 + 2(8)^2}{3} + 12(8)^2$$

$$= \frac{16384 + 3072 + 128}{3} + 768$$

$$= \frac{19584}{3} + 768$$

$$= 6528 + 768$$

$$= \mathbf{7296}$$

Again, you can also use:

$$\sum_{i=1}^n 4n(i^2 + 3)$$

$$= \sum_{i=1}^8 32(i^2 + 3)$$

Limits and the n Variable

In this subsection, we will apply what we had learned in the previous example to examine when a sum approaches a limit.

Let's take a look at an example:

Simplify the expression and find the limit as $n \rightarrow \infty$.

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{36i}{n^2}$$

We want to find the sum as n approaches infinity. Similar to what we had done in Calculus 1 when we created a table of values to find the limit of $f(x)$ as x approached a specific value, we will do the same in this example.

We will evaluate the sum where $n = 10, 100, 1000, \text{ and } 10000$.

Step 1: Rewrite the expression to find a formula of the sum of n terms.

Similar to the previous example, we will simplify the expression so that we can just plug in our values for n .

$$\begin{aligned} \sum_{i=1}^n \frac{36i}{n^2} &= \frac{36}{n^2} \sum_{i=1}^n i \\ &= \frac{36}{n^2} \left[\frac{n(n+1)}{2} \right] \\ &= \frac{36n^2 + 36n}{2n^2} \\ &= \frac{36n(n+1)}{2n^2} \\ &= \frac{18(n+1)}{n} \end{aligned}$$

Step 2: Evaluate the sum for each value of n and construct a table.

n	$\sum_{i=1}^n \frac{36i}{n^2} = \frac{18(n+1)}{n}$
10	19.8
100	18.18

1000	18.018
10 000	18.0018

When we plug in our n values into the formula:

For example,

$n = 10$:

$$= \frac{18(10 + 1)}{10}$$

$n = 100$:

$$= \frac{18(100 + 1)}{100}$$

$n = 1000$

$$= \frac{18(1000 + 1)}{1000}$$

$n = 10\,000$

$$= \frac{18(10\,000 + 1)}{10\,000}$$

We get the values in the table above. As you can notice from the table, the sum approaches 18 (the limit) as n increases to infinity.

Also note that the value of n has to be a positive integer value.

Returning to the beginning of the section, we can also evaluate the limit as follows:

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{36i}{n^2}$$

We already rewrote the expression and concluded that

$$\sum_{i=1}^n \frac{36i}{n^2} = \frac{18(n + 1)}{n}$$

Thus,

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} \frac{18(n+1)}{n} \\
 &= \lim_{n \rightarrow \infty} \frac{18n + 18}{n} \\
 &= \lim_{n \rightarrow \infty} \left(\frac{18n}{n} + \frac{18}{n} \right) \\
 &= \lim_{n \rightarrow \infty} \left(18 + \frac{18}{n} \right) \\
 &= 18 + 0 \\
 &= 18
 \end{aligned}$$

The solution matches the value from the table above. **We then conclude that the limit of $\frac{18(n+1)}{n}$ as n approaches infinity is 18.**

Area

In the beginning of Calculus 1, we looked at two important concepts in calculus: the tangent line and the area problem. In this section, we will go into more detail in regards to finding the area of a region that is located between a function.

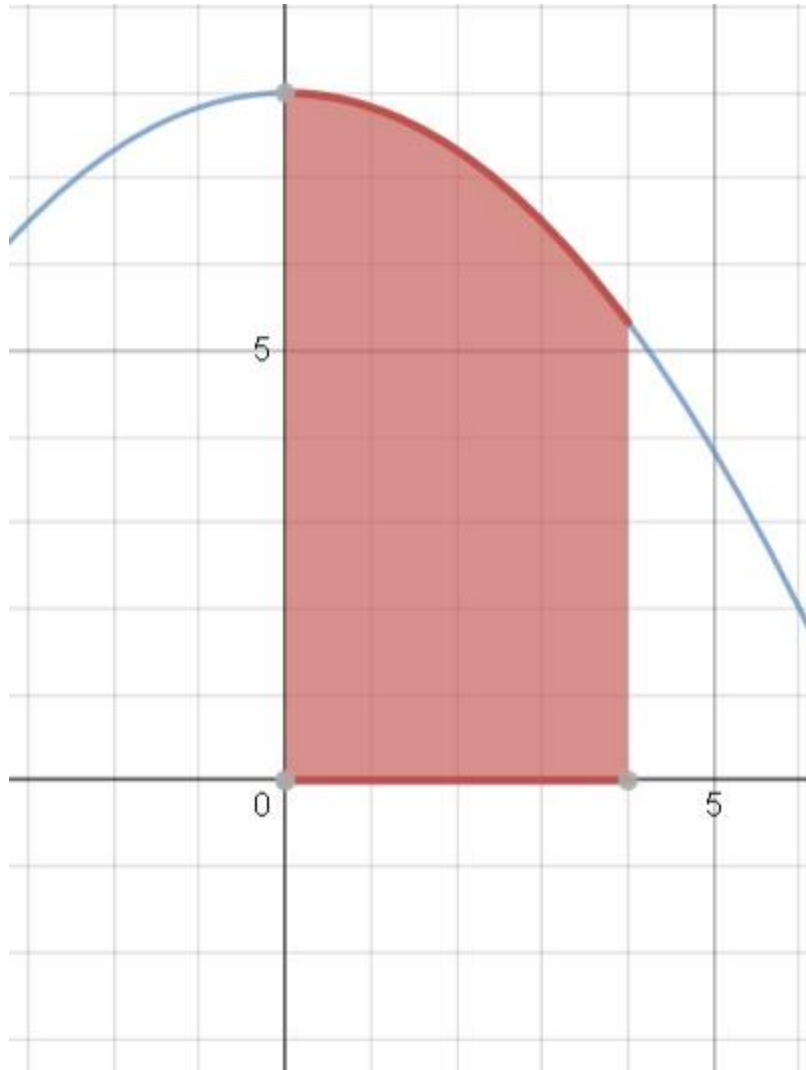
If you can recall, we used rectangles to approximate the area under the graph when we examined the area problem.

Let's take a look at an example:

Approximate the area of the region that lies between the graph of

$$f(x) = 8 - \frac{x^2}{6}$$

and the x -axis between $x = 0$ and $x = 4$ using 4 rectangles.



The region we are looking at.

Step 1: Find the interval $[a, b]$ and divide it into subintervals of equal widths.

From the given information, our interval is $[0, 4]$.

n is the number of rectangles (subdivided intervals) we want to divide it into (in this example, we want to divide it into 4 rectangles so $n = 4$)

To divide the interval into n subintervals:

$$\Delta x = \frac{b - a}{n}$$

$$\Delta x = \frac{4 - 0}{4}$$

$$\Delta x = 1$$

Using the right endpoints

Step 2a: Use the right endpoints of the four intervals we divided to find the area of the rectangles.

For the first approximation we will use the right endpoints of the rectangle to find the area.

This is where we will apply what we learned in the previous section regarding sums. We will sum the area of the rectangles to make the approximation. When you look at the graph, we start with the right endpoint at $x = 1$.

We then find the sum of the four rectangles:

$$\sum_{i=1}^n f(i) (\Delta x)$$

n will be the number of rectangles (upper limit of summation)

$f(i)$ will be the height (taken from the right endpoints)

Δx will be the width (taken from the value when we divided the interval into subinterval rectangles of equal width)

Thus we get

$$\sum_{i=1}^4 f(i)(1)$$

To find i , multiply i and Δx . This is what we will put into the x variable in our function. In our example, our Δx is 1 so we multiply $(i)(1)$ (this example is simpler so we just need to put in i).

**If we doubled the rectangles (8 instead of 4) and $\Delta x = \frac{4-0}{8} = \frac{1}{2}$, we would multiply

$(i) \left(\frac{1}{2}\right)$ and we get $\frac{i}{2}$. We will do an example and double the rectangles after I finish this main example)**

$f(i)$ is when we evaluate the function at the right endpoints (basically, you just put i into the function):

$$f(x) = 8 - \frac{x^2}{6}$$

$$f(i) = 8 - \frac{i^2}{6}$$

Since we are using 4 rectangles, the upper limit of summation is 4 ($n = 4$).

Let's find $f(i)$ first and then plug in back into $\sum_{i=1}^4 f(i)(1)$:

$$\sum_{i=1}^4 \left(8 - \frac{i^2}{6} \right)$$

Using the properties and summation formulas we learned:

$$= \sum_{i=1}^4 8 - \frac{1}{6} \sum_{i=1}^4 i^2$$

Recall:

$$\sum_{i=1}^4 8 = 8(4) = 32$$

$$\begin{aligned} \left(\frac{1}{6} \right) \left[\sum_{i=1}^4 i^2 \right] &= \left(\frac{1}{6} \right) \left[\frac{4(4+1)(2(4)+1)}{6} \right] \\ &= \left(\frac{1}{6} \right) (30) \end{aligned}$$

$$= 5$$

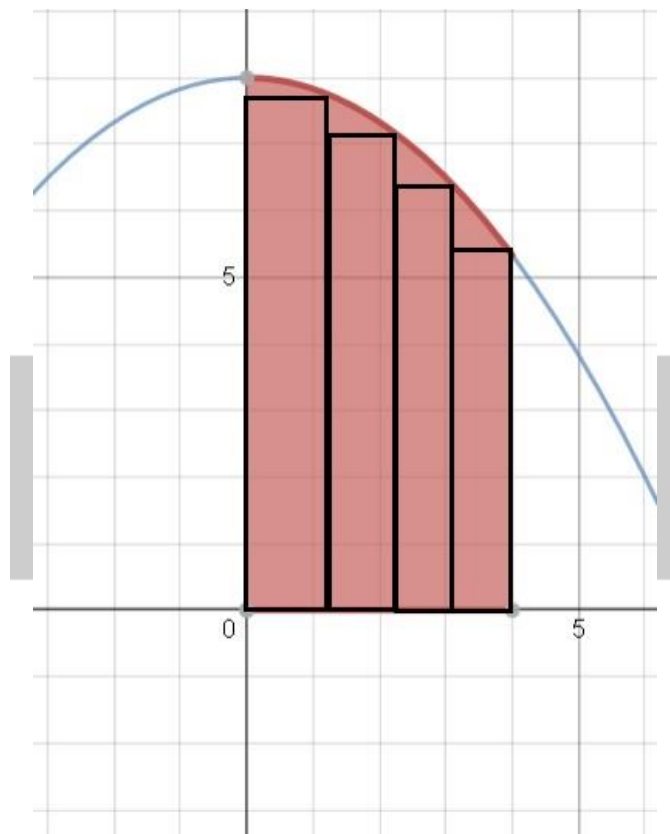
Putting everything together:

$$= \sum_{i=1}^4 8 - \frac{1}{6} \sum_{i=1}^4 i^2$$

$$= 32 - 5$$

$$= \mathbf{27}$$

The area of the four rectangles is 27. Since the rectangles are inside the region bound by the parabola, you can see from the graph that there are some areas missed by the rectangles. **We then form the conclusion that the area is greater than 27 (the area of the four rectangles).**

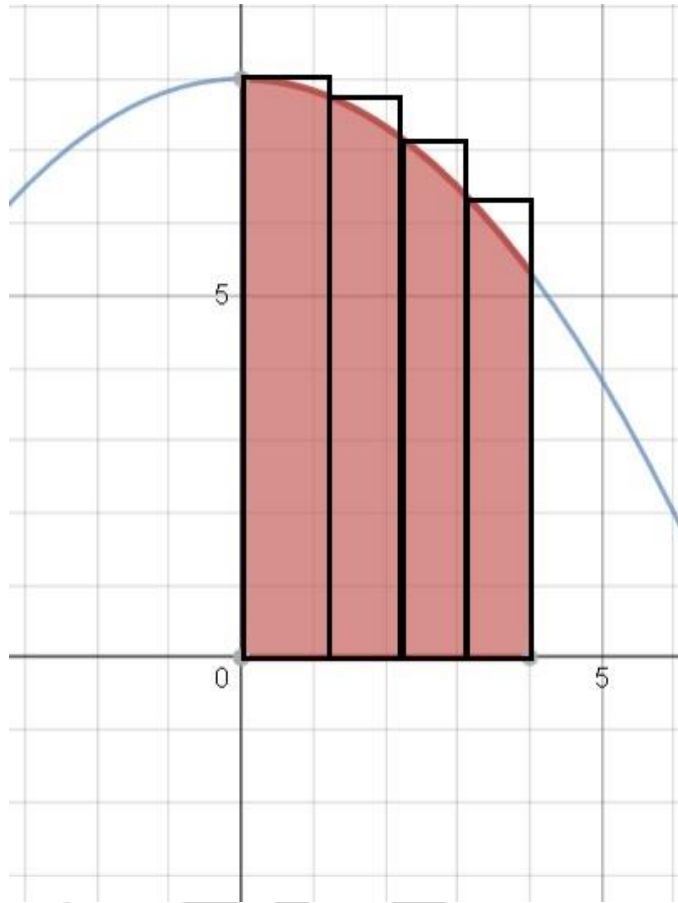


The rectangles miss some of the area of the region which is why we have to find the area using the left endpoints.

Using the left endpoints

Step 2b: Use the left endpoints of the four intervals we divided to find the area of the rectangles.

For the second approximation, we will use the left endpoints to find the area within our specified region under the parabola. As you can see from the graph, this time the rectangles extend beyond the specified region. We can conclude that the area of the rectangles will be greater than the area of the region (in contrast to Step 2a where the area of the rectangles was less than the area of the region).



The left endpoints touch the parabola this time. The area of the rectangles is now greater than the area of the shaded region.

In Step 2a, we multiplied by i . For the left endpoints, we multiply by $i - 1$ since we start at the left endpoint of the rectangle.

Since we are using the left endpoints this time we note that the left endpoints are:

$$\sum_{i=1}^4 f(i-1)(1)$$

$$= \sum_{i=1}^4 \left[8 - \frac{(i-1)^2}{6} \right] (1)$$

Again, we use our summation formulas and rules:

$$= \sum_{i=1}^4 \left[8 - \frac{(i-1)^2}{6} \right]$$

$$= \sum_{i=1}^4 8 - \frac{1}{6} \sum_{i=1}^4 (i-1)^2$$

For the first term:

$$\sum_{i=1}^4 8 = (8)(4) = 32$$

For the second term:

$$\frac{1}{6} \sum_{i=1}^4 (i-1)^2 = \frac{1}{6} \sum_{i=1}^4 (i^2 - 2i + 1)$$

$$= \frac{1}{6} \left[\sum_{i=1}^4 i^2 - \sum_{i=1}^4 2i + \sum_{i=1}^4 1 \right]$$

$$= \frac{1}{6} \left[\sum_{i=1}^4 i^2 - 2 \sum_{i=1}^4 i + \sum_{i=1}^4 1 \right]$$

For the second term (part I):

$$\sum_{i=1}^n i^2$$

$$= \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^4 i^2$$

$$= \frac{4(4+1)(2(4)+1)}{6}$$

$$= 30$$

For the second term (part II):

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$2 \sum_{i=1}^4 i = 2 \left(\frac{4(4+1)}{2} \right)$$

$$= 20$$

For the second term (part III):

$$\sum_{i=1}^n c$$

$$\sum_{i=1}^4 1 = (1)(4) = 4$$

The final answer to the second term:

$$\frac{1}{6} \sum_{i=1}^4 (i-1)^2 = \frac{1}{6} \sum_{i=1}^4 (i^2 - 2i + 1)$$

$$= \frac{1}{6} [30 - 20 + 4]$$

$$= \frac{14}{6}$$

Now we can put it together:

$$= \sum_{i=1}^4 \left[8 - \frac{(i-1)^2}{6} \right]$$

$$= \sum_{i=1}^4 8 - \frac{1}{6} \sum_{i=1}^4 (i-1)^2$$

$$= 32 - \frac{14}{6}$$

Finding a common denominator:

$$32 = \frac{192}{6}$$

So,

$$\begin{aligned} &= \frac{192}{6} - \frac{14}{6} \\ &= \frac{178}{6} = \frac{89}{3} \end{aligned}$$

The area of the four rectangles is $\frac{89}{3} \approx 29.67$

This fits with what we concluded at the beginning of this step (Step 2b). Since the area of the rectangles is greater than the area of the region bounded under the parabola, it makes sense that the area using the left endpoints is greater than the area using the right endpoints.

Step 3: Use the information from the area using the right endpoints and the area using the left endpoints to make a conclusion about the area of the specified region.

Since the area using the right endpoints is less than the region bounded under the parabola and the area using the left endpoints is greater than the region bounded under the parabola, we can conclude that the area of the specified region bounded under the parabola falls between these two values. Thus,

$$27 < \text{area of the region} < 29.67$$

If you recall from when I discussed this in Calculus 1, increasing the number of rectangles will give you a closer approximation to the area of the specified region. For our example, let's double the number of rectangles. Instead of four rectangles, let's use eight.

The width, $\Delta x = \frac{1}{2}$ since $\frac{b-a}{n} = \frac{4-0}{8} = \frac{4}{8} = \frac{1}{2}$.

For the right endpoints, we multiply by $\frac{1}{2}$ instead of 1. For the function, we plug in $\frac{1}{2}i$ (or you can simplify it and put i into the numerator, $\frac{i}{2}$). Since we are using eight rectangles, $n = 8$.

We then perform the summation using the right endpoints:

$$\sum_{i=1}^n f(i) (\Delta x)$$

$$= \left[\sum_{i=1}^8 8 - \frac{\left(\frac{i}{2}\right)^2}{6} \right] \left(\frac{1}{2}\right)$$

$$= \left[\sum_{i=1}^8 8 - \frac{i^2}{24} \right] \left(\frac{1}{2}\right)$$

When we evaluate using the right endpoints, the sum of the area of the eight rectangles is **27.75**.

We then perform the summation using the left endpoints:

$$\sum_{i=1}^8 f(i-1) \left(\frac{1}{2}\right)$$

$$= \left[\sum_{i=1}^8 8 - \frac{\left(\frac{i-1}{2}\right)^2}{6} \right] \left(\frac{1}{2}\right)$$

$$= \left[\sum_{i=1}^8 8 - \frac{i^2 - 2i + 1}{24} \right] \left(\frac{1}{2}\right)$$

$$= \left[\sum_{i=1}^8 8 - \frac{1}{24} \left(\sum_{i=1}^8 i^2 - 2 \sum_{i=1}^8 i + \sum_{i=1}^8 1 \right) \right] \left(\frac{1}{2}\right)$$

When we evaluate using the left endpoints, the sum of the area of the eight rectangles is around **29.08**.

Since the area of the specified region falls between these two areas, we get

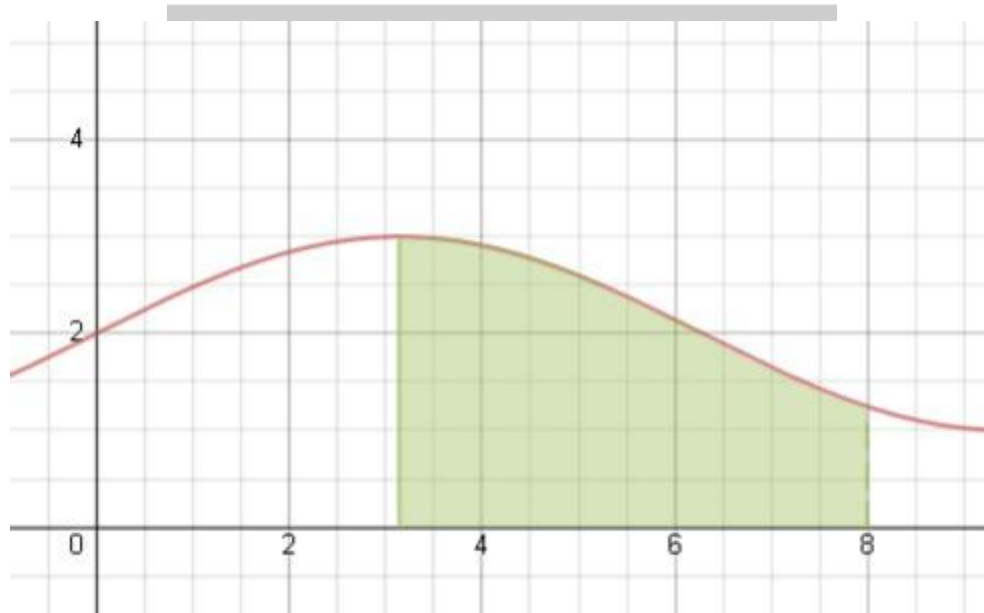
$$\mathbf{27.75 < \text{area of the region} < 29.08}$$

As you can see, we get a closer approximation of the area of the region when we **increase** the number of rectangles.

Upper and Lower Sums for a Region

When we covered the section on the applications of differentiation, we discussed the **Extreme Value Theorem (EVT)**. It basically stated that if a function is continuous on a closed interval (in our examples, we noted the interval as $[a, b]$), the function has a minimum and a maximum on the interval.

We can see this through the graph:



The area of the region is bounded by graph of f , the x -axis under it and the two vertical lines $x = 3.2$ and $x = 8$. In this graph for the function f , the interval $[a, b]$ is $[3.2, 8]$.

In this section, we will dissect the example from the previous section and go into more depth explaining the elements when finding the area of a region using rectangles.

In the previous sections, when we wanted to approximate the area of a region we divided the bounded interval under the graph into subintervals. According to EVT, since the function is continuous on the closed interval, we know that the function has a minimum value and a maximum value in each of the divided subintervals.

$f(m_i)$ is the minimum value of $f(x)$ in i th subinterval

$f(M_i)$ is the maximum value of $f(x)$ in i th subinterval

To subdivide the interval into n subintervals (where n is the number of subdivided intervals or rectangles) we used

$$\Delta x = \frac{b - a}{n}$$

to determine the width of each subinterval (the width of the rectangle).

So if we wanted to subdivide the interval $[3, 9]$ into three subintervals:

$$\Delta x = \frac{9 - 3}{3} = \frac{6}{3} = 2$$

In the previous example, we approximated the area using the left endpoints and the right endpoints. The endpoints of the intervals are noted as

$$a + 0(\Delta x) < a + 1(\Delta x) < a + 2(\Delta x) < a + 3(\Delta x) \dots < a + n(\Delta x)$$

$$a = x_0$$

$$a + 0(\Delta x)$$

$$x_4$$

$$a + 4(\Delta x)$$

$$b = x_n$$

$$a + n(\Delta x)$$

We start with $a + i(\Delta x)$. Recall when we looked at sigma notation, i is the lower limit of summation (the one we start with). If we have the following:

$$\sum_{i=2}^6 i^3$$

Here the lower limit of summation is 2 and upper limit of summation is 6.

The right endpoints of the intervals will then start with $a + 2(\Delta x)$, $a + 3(\Delta x)$, ... and continue on to $a + 6(\Delta x)$.

So if we want to find the right endpoints for the subintervals between $[3, 9]$:

We divided the subinterval and found the width: $\Delta x = 2$.

$$a + i(\Delta x)$$

$$= 3 + 2i$$

For the left endpoints:

$$a + (i - 1)(\Delta x)$$

$$= 3 + 2(i - 1)$$

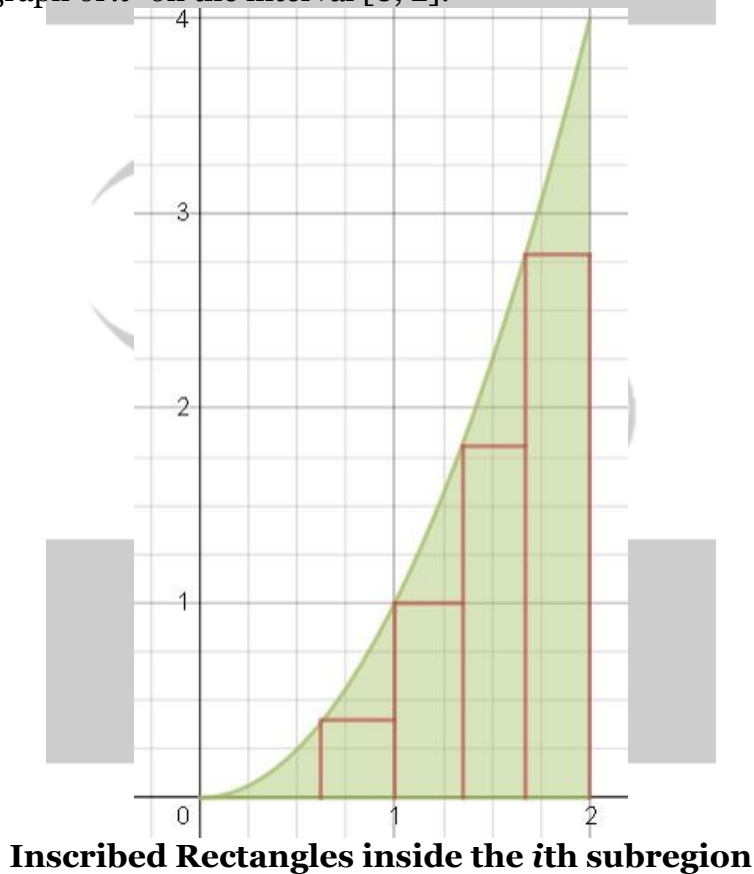
Inscribed rectangles are inside the i th subregion while circumscribed rectangles extend beyond the i th subregion. (We'll look at the diagrams in the next few paragraphs)

The area of the inscribed rectangles is less than or equal to the area of the circumscribed rectangles. In other words,

$$f(m_i)\Delta x \leq f(M_i)\Delta x$$

The **lower sum** is the sum of the areas of the inscribed rectangles. If you recall from the previous example, the areas of the inscribed rectangles did not include all of the area of the region.

Let's look at the graph of x^2 on the interval $[0, 2]$:

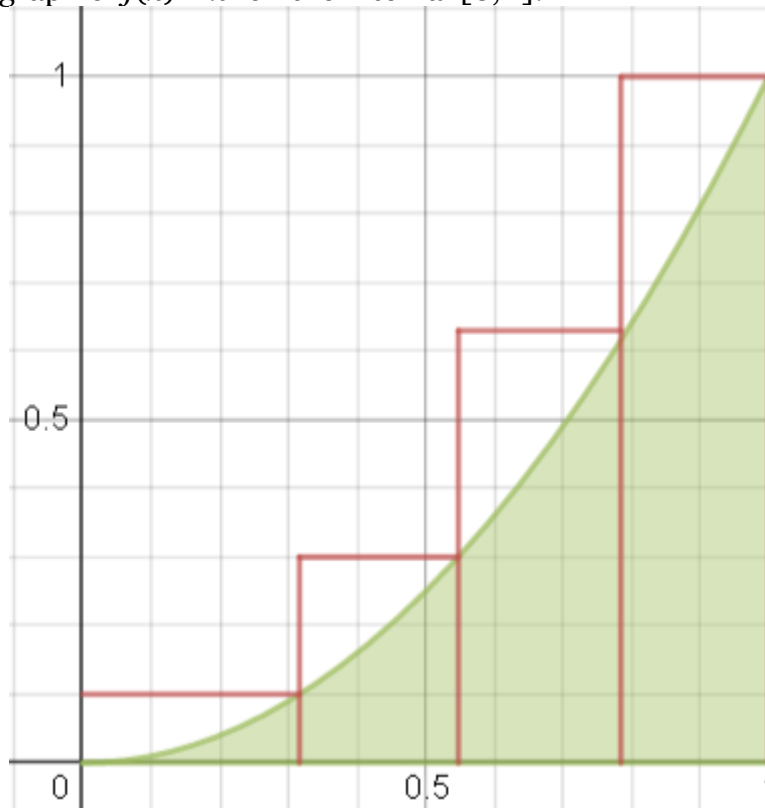


When we put everything together, the lower sum

$$s(n) = \sum_{i=1}^n f(m_i)\Delta x$$

To include more of the area, we found the areas of the circumscribed rectangles. This overestimated the area of the region since the area of the circumscribed rectangles included the area of the region and a little bit more from the rectangles which were not part of the region. The **upper sum** is the sum of the areas of the circumscribed rectangles.

Let's look at the graph of $f(x) = x^2$ on the interval $[0, 1]$:



Circumscribed Rectangles extending outside the i th subregion

When we put everything together, the upper sum

$$S(n) = \sum_{i=1}^n f(M_i) \Delta x$$

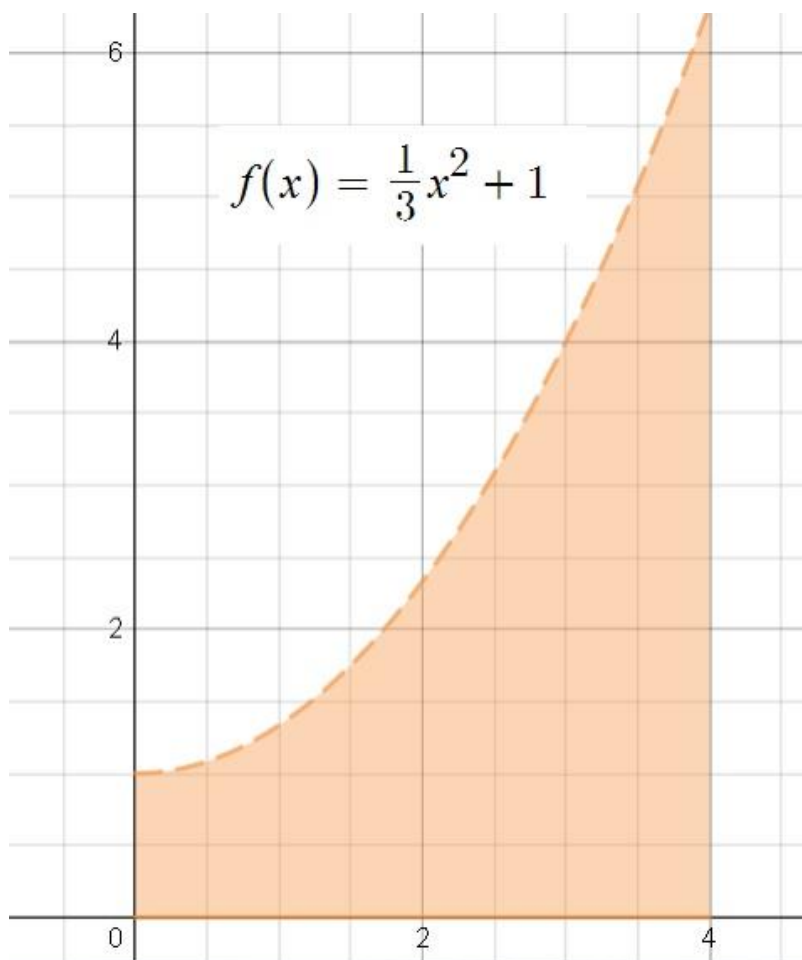
We then concluded that the area of the region fell between these two areas:

$$s(n) \leq \text{area of the region} \leq S(n)$$

Now let's do an example where we apply the general concepts that we covered here.

Example

Consider a region bounded by the graph of the function, $f(x) = \frac{1}{3}x^2 + 1$ and the x -axis between $x = 0$ and $x = 4$. Find the upper and lower sums of the region.



First, note the function, f and the interval:

$$f(x) = \frac{1}{3}x^2 + 1$$

and the interval $[a, b]$ is $[0, 4]$.

Step 1: Divide the interval into n subintervals with a width of Δx .

$$\Delta x = \frac{b - a}{n}$$

$$\Delta x = \frac{4 - 0}{n} = \frac{4}{n}$$

Step 2: Define the endpoints of the interval.

We can see from the graph that the function is increasing along the interval. The minimum value of the function will occur at the left endpoint. The maximum value will occur at the right endpoint.

The left endpoint is denoted by

$$m_i = a + (i - 1)\Delta x$$

$$m_i = 0 + (i - 1)\left(\frac{4}{n}\right)$$

$$= \frac{4(i - 1)}{n}$$

The right endpoint is denoted by

$$M_i = a + (i)\Delta x$$

$$M_i = 0 + (i)\left(\frac{4}{n}\right)$$

$$= \frac{4i}{n}$$

Step 3: Evaluate the upper and lower sums using the information from the left and right endpoints.

3a: Lower Sum

Since we saw that the minimum value occurred at the left endpoint, we use the left endpoints to evaluate the lower sum.

$$s(n) = \sum_{i=1}^n f(m_i)\Delta x$$

$$= \sum_{i=1}^n f\left(\frac{4(i - 1)}{n}\right)\left(\frac{4}{n}\right)$$

Now we put it into $f(x)$:

$$\begin{aligned}
&= \sum_{i=1}^n \left[\frac{1}{3} \left(\frac{4(i-1)}{n} \right)^2 + 1 \right] \left(\frac{4}{n} \right) \\
&= \left[\frac{1}{3} \sum_{i=1}^n \left(\frac{4}{n} \right)^2 (i-1)^2 + \sum_{i=1}^n 1 \right] \left(\frac{4}{n} \right) \\
&= \left[\frac{1}{3} \sum_{i=1}^n \left(\frac{16}{n^2} \right) (i^2 - 2i + 1) + \sum_{i=1}^n 1 \right] \left(\frac{4}{n} \right)
\end{aligned}$$

Multiply by using the distributive property $[a(b+c) = ab + ac]$:

In our case, the a would be $\frac{4}{n}$.

$$\begin{aligned}
&= \left[\frac{1}{3} \sum_{i=1}^n \left(\frac{64}{n^3} \right) (i^2 - 2i + 1) + \sum_{i=1}^n \frac{4}{n} \right] \\
&= \frac{64}{3n^3} \sum_{i=1}^n (i^2 - 2i + 1) + \frac{1}{n} \sum_{i=1}^n 4
\end{aligned}$$

Expanding inside the brackets and simplifying:

$$= \frac{64}{3n^3} \left[\sum_{i=1}^n i^2 - 2 \sum_{i=1}^n i + \sum_{i=1}^n 1 \right] + 4$$

Since we already did similar steps in the previous examples, I will just skip those steps:

$$\begin{aligned}
&= \frac{64}{3n^3} \left[\frac{2n^3 - 3n^2 + n}{6} \right] + 4 \\
&= \frac{32(2n^2 - 3n + 1)}{9n^2} + 4 \\
&= \frac{64n^2 - 96n + 32}{9n^2} + 4
\end{aligned}$$

Find common denominators to add the first part with the n :

$$= \frac{64n^2 - 96n + 32}{9n^2} + \frac{36n^2}{9n^2}$$

Then we can add everything together:

$$= \frac{64n^2 - 96n + 32 + 36n^2}{9n^2}$$

$$= \frac{100n^2 - 96n + 32}{9n^2}$$

Simplify:

$$= \frac{100n^2}{9n^2} - \frac{96n}{9n^2} + \frac{32}{9n^2}$$

$$= \frac{100}{9} - \frac{32}{3n} + \frac{32}{9n^2}$$

When we simplify, we get our lower sum:

$$s(n) = \frac{100}{9} - \frac{32}{3n} + \frac{32}{9n^2}$$

3b: Upper Sum

Since the maximum value occurred at the right endpoint, we use the right endpoints to evaluate the upper sum.

$$S(n) = \sum_{i=1}^n f(M_i) \Delta x$$

We use the same process as we did for the lower sum, but this time, we use $\frac{4i}{n}$ instead of $\frac{4(i-1)}{n}$.

$$S(n) = \sum_{i=1}^n f\left(\frac{4i}{n}\right) \left(\frac{4}{n}\right)$$

This one should be relatively easier to work with compared to the lower sum.

$$= \sum_{i=1}^n \left[\frac{1}{3} \left(\frac{4i}{n}\right)^2 + 1 \right] \left(\frac{4}{n}\right)$$

$$\begin{aligned}
&= \sum_{i=1}^n \left(\frac{4}{3n} \right) \left(\frac{16}{n^2} \right) (i^2) + \frac{4}{n} \\
&= \frac{64}{3n^3} \sum_{i=1}^n i^2 + \frac{1}{n} \sum_{i=1}^n 4 \\
&= \frac{64}{3n^3} \left[\frac{n(n+1)(2n+1)}{6} \right] + \frac{4n}{n}
\end{aligned}$$

$$= \frac{64}{3n^3} \left[\frac{2n^3 + 3n^2 + n}{6} \right] + 4$$

Simplify:

$$= \frac{32}{3n^3} \left[\frac{2n^3 + 3n^2 + n}{3} \right] + 4$$

$$= \frac{32(2n^2 + 3n + 1)}{9n^2} + 4$$

$$= \frac{32(2n^2 + 3n + 1)}{9n^2} + \frac{36n^2}{9n^2}$$

$$= \frac{64n^2 + 96n + 32 + 36n^2}{9n^2}$$

$$= \frac{100n^2 + 96n + 32}{9n^2}$$

$$= \frac{100n^2}{9n^2} + \frac{96n}{9n^2} + \frac{32}{9n^2}$$

$$= \frac{100}{9} + \frac{32}{3n} + \frac{32}{9n^2}$$

When we simplify we get our upper sum:

$$S(n) = \frac{100}{9} + \frac{32}{3n} + \frac{32}{9n^2}$$

If you recall from the beginning of this section, the value of the lower sum is less than or equal to the value of the upper sum. We can see this when we completed our example:

$$s(n) = \frac{100}{9} - \frac{32}{3n} + \frac{32}{9n^2}$$

$$S(n) = \frac{100}{9} + \frac{32}{3n} + \frac{32}{9n^2}$$

If $n = 2$, the lower sum $s(n)$:

$$s(2) = \frac{100}{9} - \frac{32}{3(2)} + \frac{32}{9(2)^2}$$

$$= \frac{100}{9} - \frac{32}{6} + \frac{32}{36}$$

$$= \frac{400}{36} - \frac{192}{36} + \frac{32}{36}$$

$$= \frac{20}{3}$$

If $n = 2$, the upper sum $S(n)$:

$$S(2) = \frac{100}{9} + \frac{32}{3(2)} + \frac{32}{9(2)^2}$$

$$= \frac{100}{9} + \frac{32}{6} + \frac{32}{36}$$

$$= \frac{400}{36} + \frac{192}{36} + \frac{32}{36}$$

$$= \frac{52}{3}$$

$$s(n) < S(n)$$

$$\frac{20}{3} < \frac{52}{3}$$

However, as the limit of n approaches infinity, the limits of the upper and lower sum are equal to each other. We looked at this at the end of the first section (Limits and the n Variable). When we take the limits of both the upper and lower sums,

$$\lim_{n \rightarrow \infty} s(n) = \lim_{n \rightarrow \infty} \frac{100}{9} - \frac{32}{3n} + \frac{32}{9n^2}$$

$$= \frac{100}{9} - 0 + 0$$

$$= \frac{100}{9}$$

$$\lim_{n \rightarrow \infty} S(n) = \lim_{n \rightarrow \infty} \frac{100}{9} + \frac{32}{3n} + \frac{32}{9n^2}$$

$$= \frac{100}{9} + 0 + 0$$

$$= \frac{100}{9}$$

They both approach the same limit, $\frac{100}{9}$.

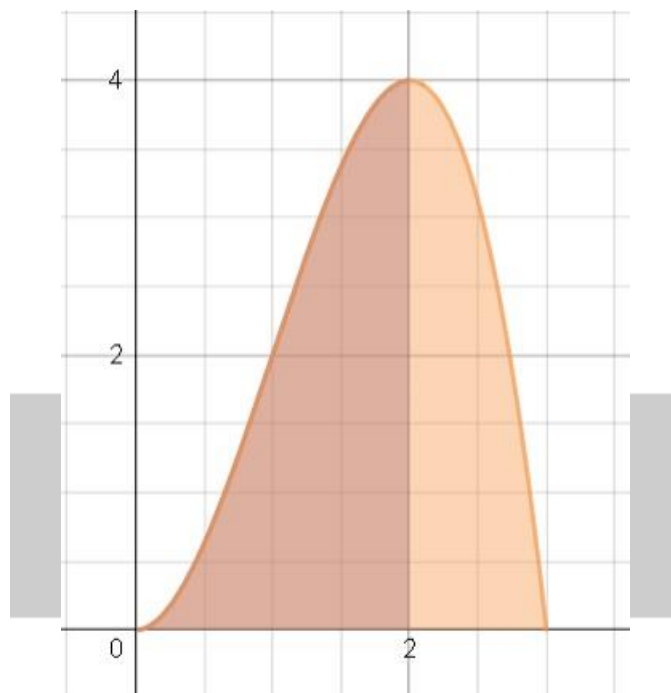
Area by the Limit Definition

For the following examples, let f be continuous and nonnegative on the interval $[a, b]$.

Example 1

Find the area of the region bounded by the graph $f(x) = 3x^2 - x^3$ and the vertical lines $x = 0$ and $x = 2$.

Here is the region we are looking at (the shaded region):



First let's divide the interval $[0, 2]$ into n subintervals:

$$\begin{aligned}\Delta x &= \frac{b-a}{n} \\ &= \frac{2-0}{n} \\ &= \frac{2}{n}\end{aligned}$$

We will perform similar steps to what we've been doing in our previous examples. As we saw at the end of the previous example, the upper and lower sums are equal to each other as n approached infinity (that is, the limit is the same when we found the minimum value $f(m_i)$ and the maximum value $f(M_i)$). We saw in the previous example that the area of the region was $\frac{100}{9}$. This brings back memories of the Squeeze Theorem!

If you recall from the Squeeze Theorem,

Let's assume

$$h(x) \leq f(x) \leq g(x)$$

for all x in an open interval containing c (except possibly at c itself) and if

$$\lim_{x \rightarrow c} h(x) = L = \lim_{x \rightarrow c} g(x)$$

then

$$\lim_{x \rightarrow c} f(x) = L$$

exists.

If we apply this to our previous example,

$$\lim_{n \rightarrow \infty} s(n) = L = \lim_{n \rightarrow \infty} S(n)$$

$$\lim_{n \rightarrow \infty} s(n) = \frac{100}{9} = \lim_{n \rightarrow \infty} S(n)$$

then

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x, \quad x_{i-1} \leq c_i \leq x_i$$

The area of the region will equal the limit above. The value for x_{i-1} is the left endpoint and the value of x_i is the right endpoint.

Also note the following:

1. f is continuous on the interval $[a, b]$
2. f is nonnegative on the interval $[a, b]$
3. the area is bounded by the x -axis, the graph of f , and the two vertical lines $x = a$ and $x = b$
4. $\Delta x = \frac{b-a}{n}$

Therefore, we can use any value of x in the i th subinterval (either the left or the right endpoints) and the value of the area of the region will be the same.

Let's use the right endpoints since it's a bit easier when performing the algebra by hand.

Step 1: Use the right endpoints (noting a choice of any x -value in the i th subinterval).

$$c_i = a + (i)\Delta x$$

$$c_i = 0 + (i)\left(\frac{2}{n}\right)$$

$$= \frac{2i}{n}$$

Step 2: Find the Area of the Region.

We know that $f(x) = 3x^2 - x^3$ and $c_i = \frac{2i}{n}$.

$$\begin{aligned} \text{Area of the region} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x \\ \text{Area of the region} &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(\frac{2i}{n}\right) \left(\frac{2}{n}\right) \end{aligned}$$

We perform the same process similar to when we found the upper sum, $S(n)$.

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \left[\sum_{i=1}^n 3 \left(\frac{2i}{n}\right)^2 - \left(\frac{2i}{n}\right)^3 \right] \left(\frac{2}{n}\right) \\ &= \lim_{n \rightarrow \infty} \left[3 \sum_{i=1}^n \frac{8i^2}{n^3} - \sum_{i=1}^n \frac{16i^3}{n^4} \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{24}{n^3} \sum_{i=1}^n i^2 - \frac{16}{n^4} \sum_{i=1}^n i^3 \right] \\ &= \lim_{n \rightarrow \infty} \left[\left(\frac{24}{n^3}\right) \left(\frac{n(n+1)(2n+1)}{6}\right) - \left(\frac{16}{n^4}\right) \left(\frac{n^2(n+1)^2}{4}\right) \right] \\ &= \lim_{n \rightarrow \infty} \left[\left(\frac{24}{n^3}\right) \left(\frac{2n^3 + 3n^2 + n}{6}\right) - \left(\frac{16}{n^4}\right) \left(\frac{n^4 + 2n^3 + n^2}{4}\right) \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{24(2n^3 + 3n^2 + n)}{6n^3} - \frac{16(n^4 + 2n^3 + n^2)}{4n^4} \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{48n^3 + 72n^2 + 24n}{6n^3} - \frac{16n^4 + 32n^3 + 16n^2}{4n^4} \right] \end{aligned}$$

$$= \lim_{n \rightarrow \infty} \left[\frac{192n^4 + 288n^3 + 96n^2}{24n^4} - \frac{96n^4 + 192n^3 + 96n^2}{24n^4} \right]$$

$$= \lim_{n \rightarrow \infty} \left[\frac{192n^4 + 288n^3 + 96n^2 - 96n^4 - 192n^3 - 96n^2}{24n^4} \right]$$

$$= \lim_{n \rightarrow \infty} \left[\frac{96n^4 + 96n^3}{24n^4} \right]$$

$$= \lim_{n \rightarrow \infty} \frac{96n^4}{24n^4} + \frac{96n^3}{24n^4}$$

$$= \lim_{n \rightarrow \infty} 4 + \frac{4}{n}$$

$$= 4 + 0$$

$$= 4$$

The area of the region is 4.

Region Bounded by y-axis

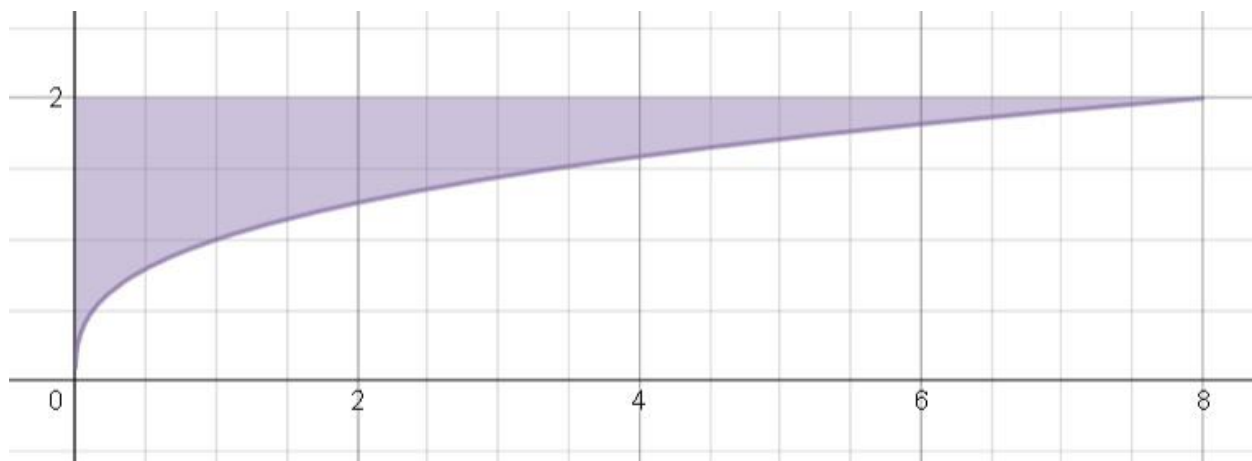
We finish up this handout by finding the area of a region bounded by the y -axis instead of the x -axis.

So far the regions in our examples were bounded by the x -axis. Now we will look at the regions bounded by the y -axis. The good thing is that we just perform the same steps that we did when the regions were bounded by the x -axis.

Example

Find the area of the region between the graph of $f(y) = y^3$ and the y -axis over the interval $0 \leq y \leq 2$.

Here is the region we will be looking at:



Step 1: Divide the interval into n subintervals.

We know that the interval is $[0, 2]$.

$$\begin{aligned}\Delta y &= \frac{b - a}{n} \\ &= \frac{2 - 0}{n} = \frac{2}{n}\end{aligned}$$

Note that instead of Δx , we use Δy .

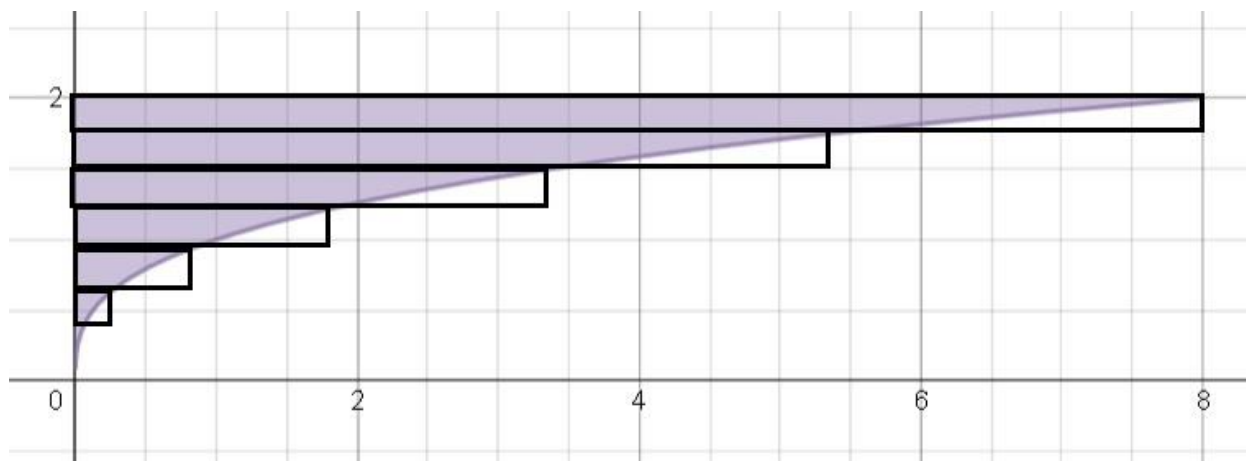
Step 2: Find the endpoints c_i .

To find the upper endpoints,

$$c_i = a + i(\Delta y)$$

$$= 0 + i\left(\frac{2}{n}\right)$$

$$= \frac{2i}{n}$$



Note that the upper endpoints are touching the graph. Instead of left and right endpoints, we look at the upper and lower endpoints when the region is bounded by the y -axis.

Step 3: Use the Limit Definition to find the Area.

Again, we use the limit definition of the area of a region and use Δy instead of Δx .

$$\text{Area of the region} = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta y$$

$$\text{Area of the region} = \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(\frac{2i}{n}\right) \left(\frac{2}{n}\right)$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{2i}{n}\right)^3 \left(\frac{2}{n}\right)$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\left(\frac{2}{n}\right)^3 (i)^3 \left(\frac{2}{n}\right) \right]$$

$$= \lim_{n \rightarrow \infty} \frac{16}{n^4} \sum_{i=1}^n i^3$$

$$= \lim_{n \rightarrow \infty} \left(\frac{16}{n^4}\right) \left[\frac{n^2(n+1)^2}{4} \right]$$

$$= \lim_{n \rightarrow \infty} \left(\frac{16}{n^4}\right) \left[\frac{n^4 + 2n^3 + n^2}{4} \right]$$

$$= \lim_{n \rightarrow \infty} \left(\frac{4(n^4 + 2n^3 + n^2)}{n^4} \right)$$

$$= \lim_{n \rightarrow \infty} \left(\frac{4n^4 + 8n^3 + 4n^2}{n^4} \right)$$

$$= \lim_{n \rightarrow \infty} \left(\frac{4n^4}{n^4} + \frac{8n^3}{n^4} + \frac{4n^2}{n^4} \right)$$

$$= \lim_{n \rightarrow \infty} \left(4 + \frac{8}{n} + \frac{4}{n^2} \right)$$

$$= 4 + 0 + 0$$

$$= 4$$

The area of the region is 4.