Calculus 1 Review
Handout 1.2
Points of Intersection and Linear Equations

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Point of Intersection

The **point of intersection** of the graphs of two equations satisfies both equations. In order to find the point (or points) of intersection, simultaneously solve the equations. You can then check your results when you graph them simultaneously.

<u>Example 1.2a</u>

Find the point of intersection for the following two lines:

$x + y = 11$
$3x - 4y = 12$

Step 1: Solve the equations for *y*.

Our two equations are in general form.

The standard form of the equation is $Ax + By = C$. We will cover this in more detail later in this handout.

For the first equation we get: $y = 11 - x$
For the second equation we get: $y = \frac{3}{4}x - 3$

Now we can equate the y-values.

Step 2: Equate the y-values and solve for x.

$11 - x = \frac{3}{4}x - 3$

$14 = \frac{7}{4}x$

$x = \frac{14}{7/4}$

$x = 8$

Now that we have the value for x, we substitute it into either equation.

Step 3: Substitute the value of x in either equation and solve for y.

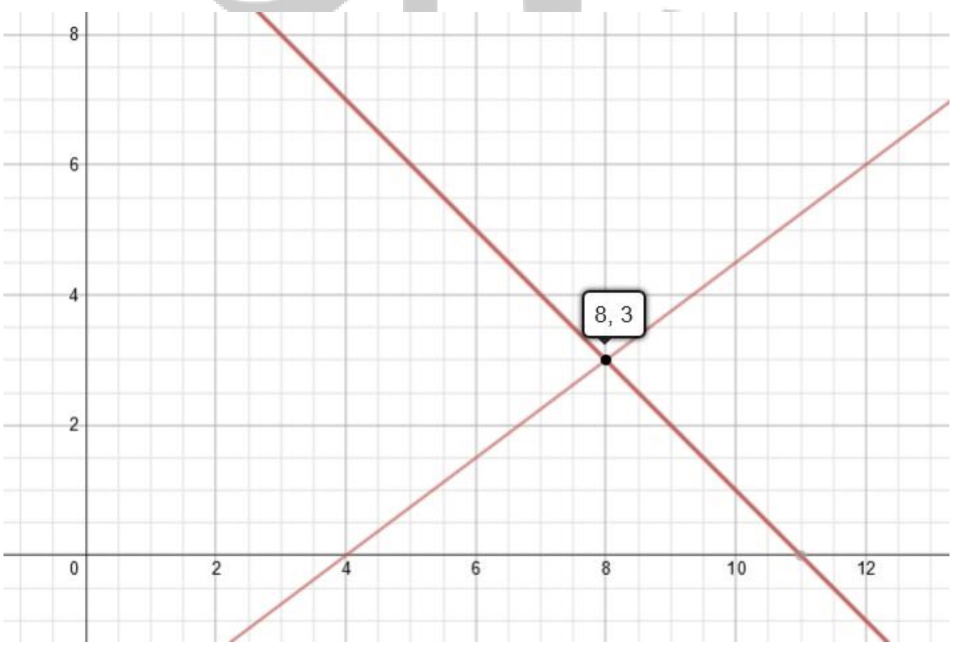
When we substitute x = 8 for either equation, we get

$8 + y = 11$
$24 - 4y = 12$

When we solve for y, ***$y = 3$***.

The point of intersection is (8, 3).

You can verify by graphing both equations simultaneously with a graphing utility.



Special thanks to Desmos

We can also find a point of intersection with nonlinear equations. Let's take a look through the next example.

Example 1.2b

Find the points of intersection for the following two equations:

$6x^2 - 2y = -10$
$8x + 4y = 40$

We follow the same steps above.

Step 1: Solve the equations for y.

First equation: $y = 5 + 3x^2$
Second equation: $y = -2x + 10$

Step 2: Equate the y-values and solve for x.

$5 + 3x^2 = -2x + 10$
We write the equation in general form.

I added 2x and subtracted 10 to both sides of the equation. The right side cancels out and equals 0.

$$\begin{array}{llll} 5 + 3x^2 & & = & -2x \quad + 10 \\ & {\color{red}+ 2x \quad - 10} & & {\color{red}+2x \quad - 10} \end{array}$$

(Remember: The general form of a quadratic is $ax^2 + bx + c = 0$)

$3x^2 + 2x - 5 = 0$

We can factor this and solve for x.

$(3x + 5)(x - 1) = 0$

$x = -\frac{5}{3}$ or $x = 1$

Step 3: Substitute the value of x in either equation and solve for y.

Let's look at the easier one first. When we substitute $x = 1$ for either equation, $y = 8$.

The point of intersection is (1, 8).

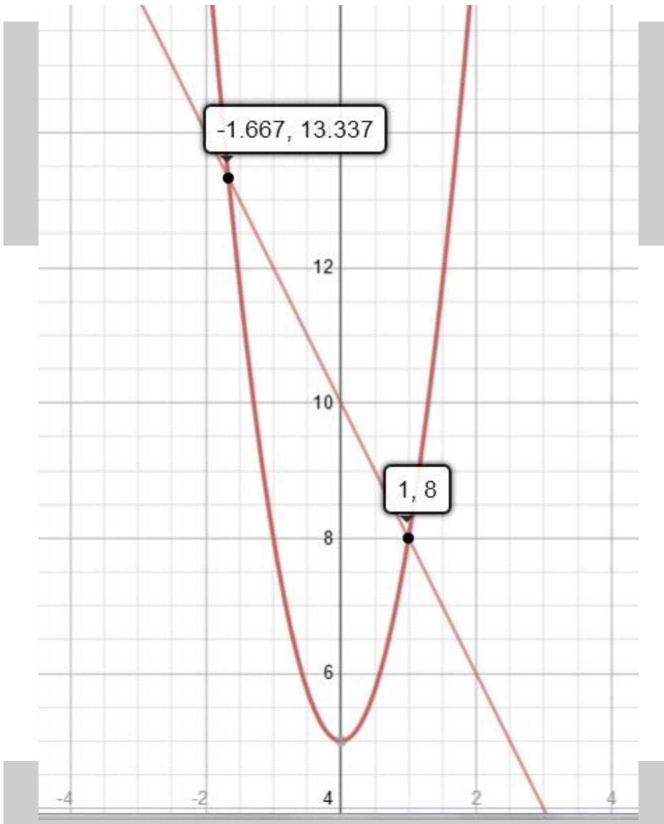
When we substitute $x = -\frac{5}{3}$ for either equation, $y = \frac{40}{3}$.

$-\frac{5}{3} \approx$ -1.667 and $\frac{40}{3} \approx$ 13.333

The point of intersection is $\left(-\frac{5}{3}, \frac{40}{3}\right)$.

So we have two points of intersection: $\left(-\frac{5}{3}, \frac{40}{3}\right)$ and **(1, 8)**.

Again, you can verify by graphing both equations simultaneously on a graphing utility.



Points of Intersection for Example 1.2b
Special thanks to Desmos.

Linear Equations

Slope of a Line

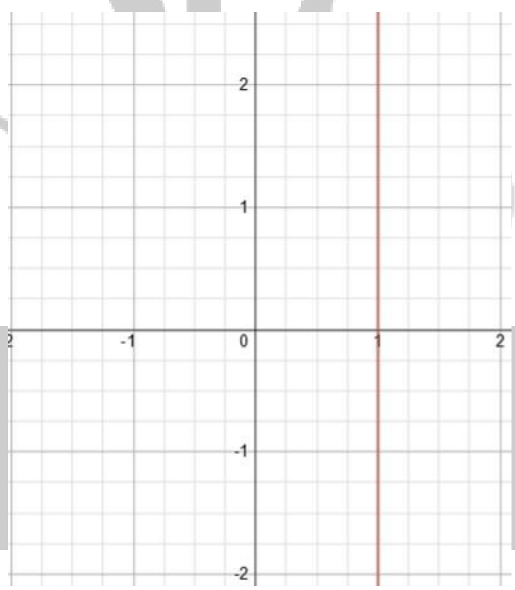
To find the slope of a nonvertical line ***m*** that passes through the points (x_1, y_1) and (x_2, y_2):

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}, \; x_1 \neq x_2$$

where the **vertical change (rise)** is denoted by: $\Delta y = y_2 - y_1$
and the **horizontal change (run)** is denoted by: $\Delta x = x_2 - x_1$

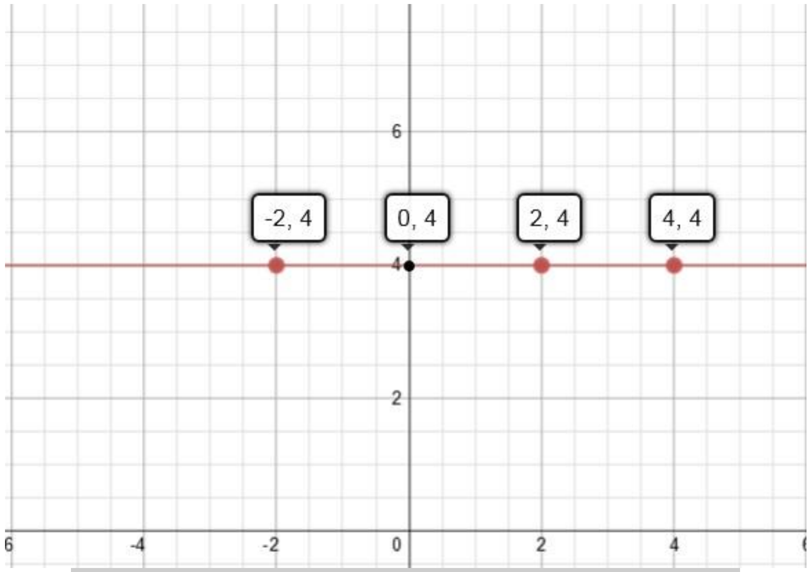
For vertical lines, the slope is undefined.
The equation of a vertical line is $\boldsymbol{x} = \mathbf{a}$, where **a** is a real number.

For example, the equation of the line $\boldsymbol{x} = \mathbf{1}$ is a vertical line whose slope ***m*** is undefined.



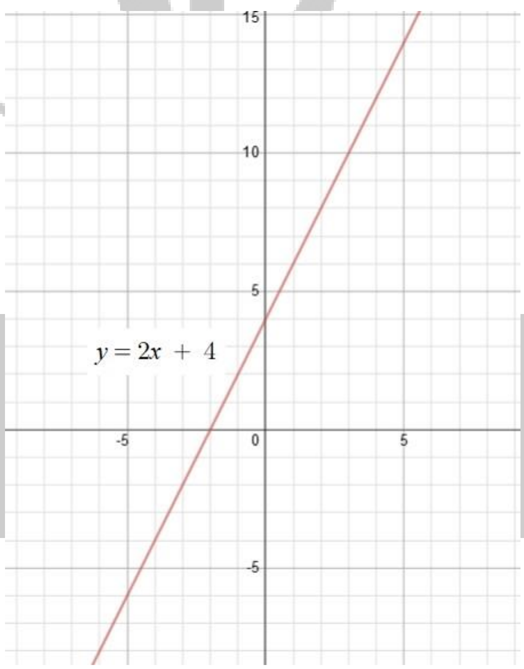
Graph for x = 1
m is undefined
Points included in the line are (1, -1), (1, 0), (1,1)... and so on

For a horizontal line, the slope is zero (0).
The equation of a horizontal line is **y = b**, where **b** is a real number.



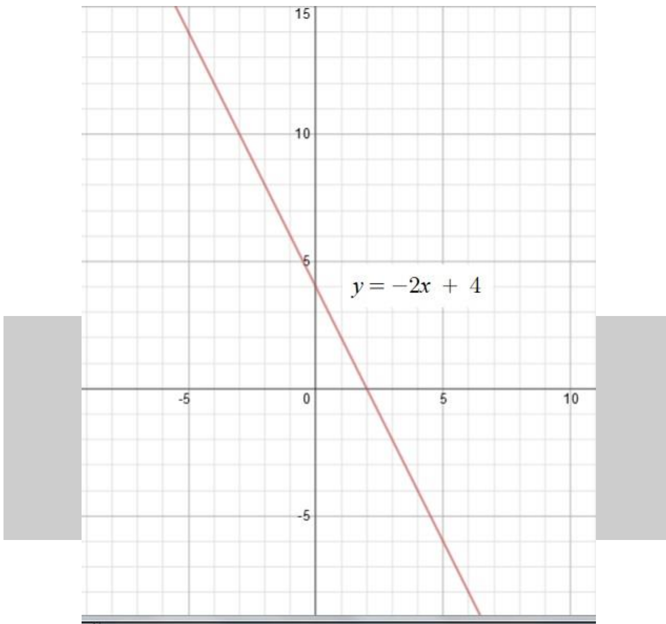
Graph for y = 4
$m = 0$
Points included in the line are (-2, 4), (0, 4), (2, 4)... and so on

For a positive slope ***m***, the line rises from left to right.



Graph of $y = 2x + 4$
$m = 2$

For a negative slope ***-m***, the line falls from left to right.

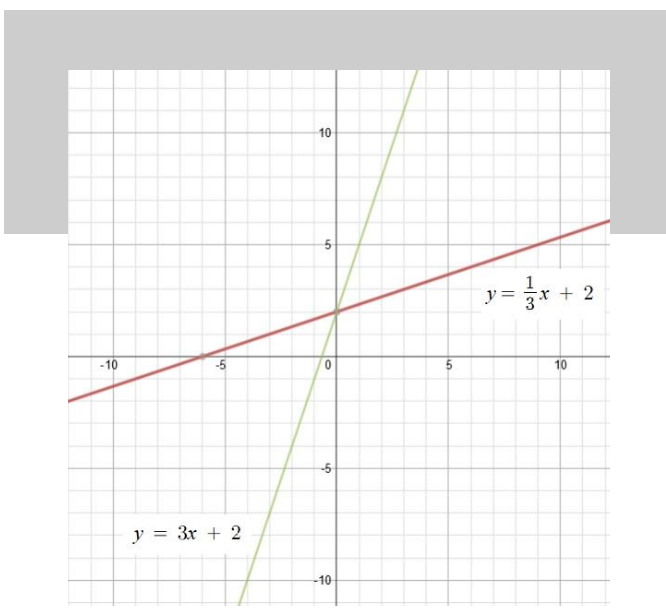


Graph of y = -2x + 4
m = -2

As the absolute value of the slope of a line increases, the line becomes steeper.

For example, graphing the two linear equations and noting their slopes and points they pass through,

$y = \frac{1}{3}x + 2$	$m = \frac{1}{3}$	(0, 2), (6, 4)
$y = 3x + 2$	$m = 3$	(0, 2), (1, 5)



We can see that the line for the second linear equation, $y = 3x + 2$ (green color) whose slope $m = 3$ is steeper than the first linear equation, $y = \frac{1}{3}x + 2$ (red color) whose slope $m = \frac{1}{3}$.

Point-slope form of a linear equation

When given the coordinates of one point (x_1, y_1) and the slope of a line m, you can write the equation of a line.

The point-slope form of a linear equation is expressed as

$$y - y_1 = m(x - x_1)$$

where (x_1, y_1) is a coordinate of a point that the line passes through and m is the slope of the line.

Please note that since vertical lines do not have a slope, they can't be written in point-slope form. If a vertical line passes through the coordinate point (2, 4), the equation will be x = 2 (from x = a).

Example 1.2c

Find the equation of the line passing through the given point (1, 5) with the slope m = 2.

We're given the following information:

Coordinate point	(1, 5)
Slope	2

Thus, we can use point-slope form to find the equation.

$y - y_1 = m(x - x_1)$

We then substitute them into the equation:
$y - 5 = 2(x - 1)$

We then perform the algebra:

$y - 5 = 2x - 2$ Multiply the 2 with the (x - 1)

$y = 2x + 3$ Add 5 to both sides to solve for y

Then we get our result. The equation of the line that passes through the point (1, 5) and has a slope of 2 is ***y = 2x + 3***.

Slope-Intercept Form of a Linear Equation

From our last example in *1.2c*, the final equation of the line that was written (*y = 2x + 3*) was in slope-intercept form.

The slope-intercept form of a linear equation

y = mx + b

where *m* is the slope and *b* is the y-intercept. If you remember from our previous handout, **to get the y-intercept, we substitute 0 for x and solve for y**.

b is the y-intercept at the point (0, b) where (0, b) is the point where the graph of the equation crosses the y-axis.

With the linear equation written in slope-intercept form, we can sketch a graph of it.

Example 1.2d

Given the following information, find the equation of the line and write it in slope-intercept form.

x-intercept	(4, 0)
y-intercept	(0, 8)

Now we will put everything we've learned into practice.

Step 1: Find the slope *m*.

$$\frac{y_2-y_1}{x_2-x_1} = \frac{8-0}{0-4} = -2$$

m = -2

Step 2: Since we have the slope and a coordinate point, we use point-slope form. (You can use either point, you will get the same equation in the end)

$y - y_1 = m(x - x_1)$

$y - 0 = -2(x - 4)$ *or* $y - 8 = -2(x - 0)$

y = -2x + 8

We get our result. The equation of the line that passes through the points (4, 0) and (0, 8) with a slope of 2 is ***$y = -2x + 8$***.

<u>General (or Standard) Form of a Linear Equation</u>

So far we have covered examples of linear equations that have slopes. However, we saw that vertical lines have an undefined slope. Since the slope is undefined we can't use point-slope form or slope-intercept form to write the equation of a vertical line.

We can however write the equation of **any line** in standard form.

The standard form of a linear equation:

$\mathbf{A}x + \mathbf{B}y = \mathbf{C}$

where A, B, and C are integers (whole numbers) and A and B do not both equal 0 ($A \neq 0$ and $B \neq 0$).

<u>Example 1.2e</u>

The following equation is written in standard form:

$3x + 4y = 12$

$A = 3$
$B = 4$
$C = 12$

If both A and B equal zero, we get $(0)x + (0)y = 12$ and $0 \neq 12$

If A = 0, we have a horizontal line:

$(\mathbf{0})x + 4y = 12$
$4y = 12$
$\mathbf{y = 3}$

If B = 0, we have a vertical line:

$3x + (\mathbf{0})y = 12$
$3x = 12$
$\mathbf{x = 4}$

Example 1.2f

Write the equation of a line that passes through the point (1, 5) with a slope of $\frac{3}{2}$ in standard form.

Step 1: Use point-slope form to solve for y.

$y - y_1 = m(x - x_1)$
$y - \mathbf{5} = \frac{\mathbf{3}}{\mathbf{2}}(x - \mathbf{1})$

$y - 5 = \frac{3}{2}x - \frac{3}{2}$

$y = \frac{3}{2}x + \frac{7}{2}$

Step 2: Bring the variables together to form Ax + By = C.

In our example, I subtracted the $\frac{3}{2}x$ from both sides. Since some people like to standardize to keep A positive, you can multiply everything by -1 if your A is negative.

$-\frac{3}{2}x + y = \frac{7}{2}$

Multiply by -1 and we get:

$\frac{3}{2}x - y = -\frac{7}{2}$

Step 3: Multiply the equation with the greatest common factor to get integer coefficients.

The greatest common factor is the largest whole number that divides evenly for each number. We multiply the numerator of the fractions with the largest number that gives us integer coefficients.

If our example was $\frac{3}{4}x - y = -\frac{5}{2}$, we can multiply everything by 4 to get

3x - 4y = -10

If we multiply everything by 2 instead, Ax will remain as a fraction. A = $\frac{6}{4} = \frac{3}{2}$. But if we multiply by 4, we get whole numbers.

Back to our example. In our example, it is 2. If we multiply everything by 2, we get whole numbers for the coefficients.

$$(\mathbf{2})\frac{3}{2}x - (\mathbf{2})y = (\mathbf{2})-\frac{7}{2}$$

The standard form of the equation is **$3x - 2y = -7$**.

The standard form of a linear equation will come in handy when we cover parallel and perpendicular lines.

Here is a review of what we covered as a reference guide:

<u>Review of Linear Equations</u>

Standard (General) form:	$\mathbf{A}x + \mathbf{B}y = \mathbf{C}$
Vertical line:	$x = a$
Horizontal line:	$y = b$
Point-slope form:	$y - y_1 = m(x - x_1)$
Slope-intercept form:	$y = mx + b$

<u>Perpendicular and Parallel Lines</u>

Now that we have a solid foundation of forms of linear equations, we can apply them to find parallel and perpendicular lines.

Two nonvertical lines are parallel if and only if their slopes are equal,

if and only if, $m_1 = m_2$

<u>Example 1.2g</u>

Find the equation of the line that passes through the point (3, 4) and is parallel to the line $2x + 6y = 12$.

We learned that two nonvertical lines are parallel if and only if $m_1 = m_2$

Step 1: Write the given equation in slope-intercept form.

The given equation is $2x + 6y = 12$

When we solve for y, $y = -\frac{1}{3}x + 2$

Step 2: Use point-slope form to find the equation of the parallel line.

Since the slope for our given equation is $-\frac{1}{3}$, $(m = -\frac{1}{3})$ and we know that the slopes are equal for parallel lines, we use point-slope form to write its equation.

$y - y_1 = m(x - x_1)$

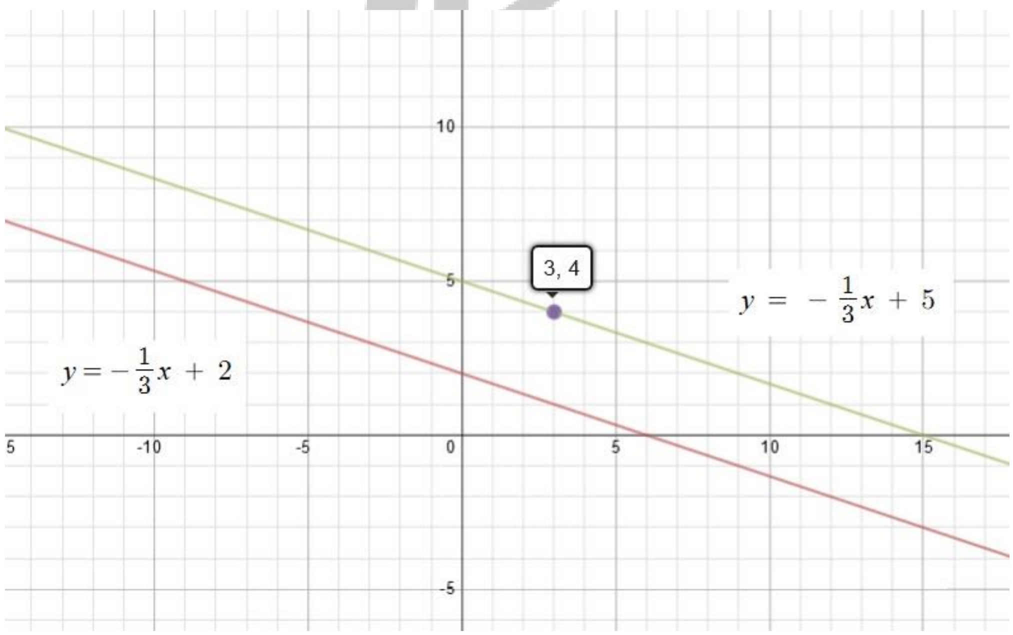
The point that the parallel line passes through is (3, 4) and the slopes are equal so $m = -\frac{1}{3}$

$\boldsymbol{y - 4 = -\frac{1}{3}(x - 3)}$

The equation of the line parallel to $2x + 6y = 12$ is

$\boldsymbol{y = -\frac{1}{3}x + 5}$

When we graph the equations, we see that they are parallel to each other and the line passes through the point (3, 4).



Two nonvertical lines are perpendicular if and only if the slopes of each line are negative reciprocals of each other,

if and only if, $m_1 = -\dfrac{1}{m_2}$

Let's use the example from **1.2g** and instead find the line that is perpendicular to the given equation.

<u>Example 1.2h</u>

Find the equation of the line that passes through the point (3, 4) and is perpendicular to the line $2x + 6y = 12$.

Step 1: Write the equation in slope intercept form.

Again, we do the same thing as we did when we were trying to find the equation of the parallel line.

The given equation is $2x + 6y = 12$

When we solve for y, $\boldsymbol{y = -\frac{1}{3}x + 2}$

However, this time we have to find the negative reciprocal of the slope. We learned that two nonvertical lines are perpendicular to each other if and only if $\boldsymbol{m_1 = -\frac{1}{m_2}}$

Since $\boldsymbol{m_1} = -\frac{1}{3}$, the slope of the perpendicular line is $\boldsymbol{-\left(\frac{1}{-\frac{1}{3}}\right)}$ which will give us - (-3).

The perpendicular slope is 3.

Step 2: Use point-slope form to find the equation of the perpendicular line.

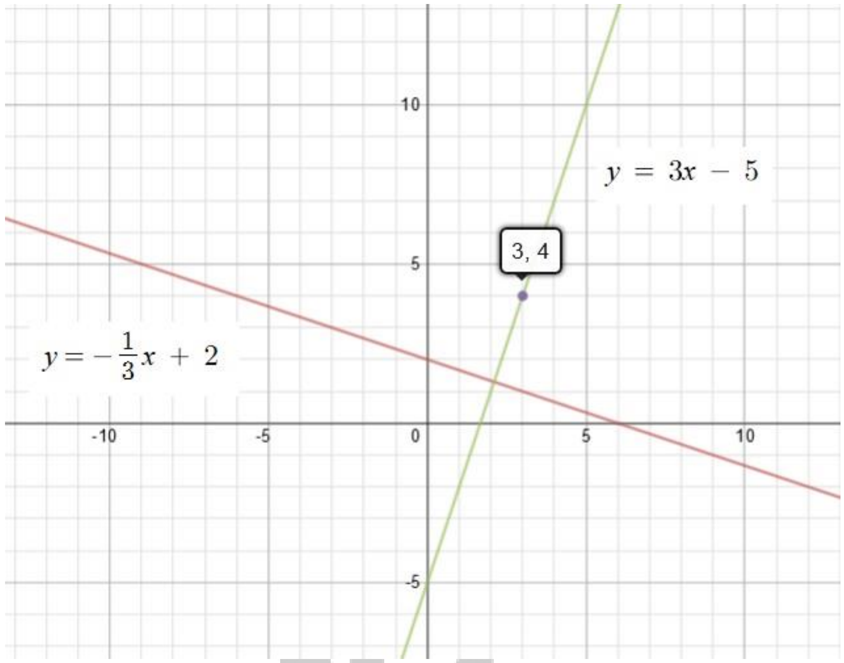
Again, we use:
$y - y_1 = m(x - x_1)$

The point that the perpendicular line passes through is (3, 4) and the slope is the negative reciprocal of the given equation so
$m = 3$

$\boldsymbol{y - 4 = 3(x - 3)}$
$\boldsymbol{y - 4 = 3x - 9}$
$\boldsymbol{y = 3x - 5}$

The equation of the line perpendicular to $2x + 6y = 12$ is

$y = 3x - 5$



When we graph the equations, we see that they are perpendicular to each other and the line passes through the point (3, 4).