Calculus I Review
Handout 1.3
Introduction to Calculus - Limits

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We are now going to dive into Calculus I as we take a look at the limit process. While precalculus covered more static applications, calculus examines dynamic applications. As such, **calculus** is the branch of mathematics that studies change.

For example, recall when we found the slope of a straight line in precalculus. We calculated the slope of the line by taking any two coordinate points along the line. Regardless of which points were chosen on the line, our slope was the same. However, with calculus we can find the slope of a curve. Unlike the slope of a straight line which exhibits a constant rate of change, the slope of the tangent line to a curve [$f'(c)$] has a non-constant rate of change which will be the derivative of the function (change in y with respect to x) at the point of tangency (at $x = c$).

We will begin by examining two concepts in calculus: **the tangent line** and **the area problem**.

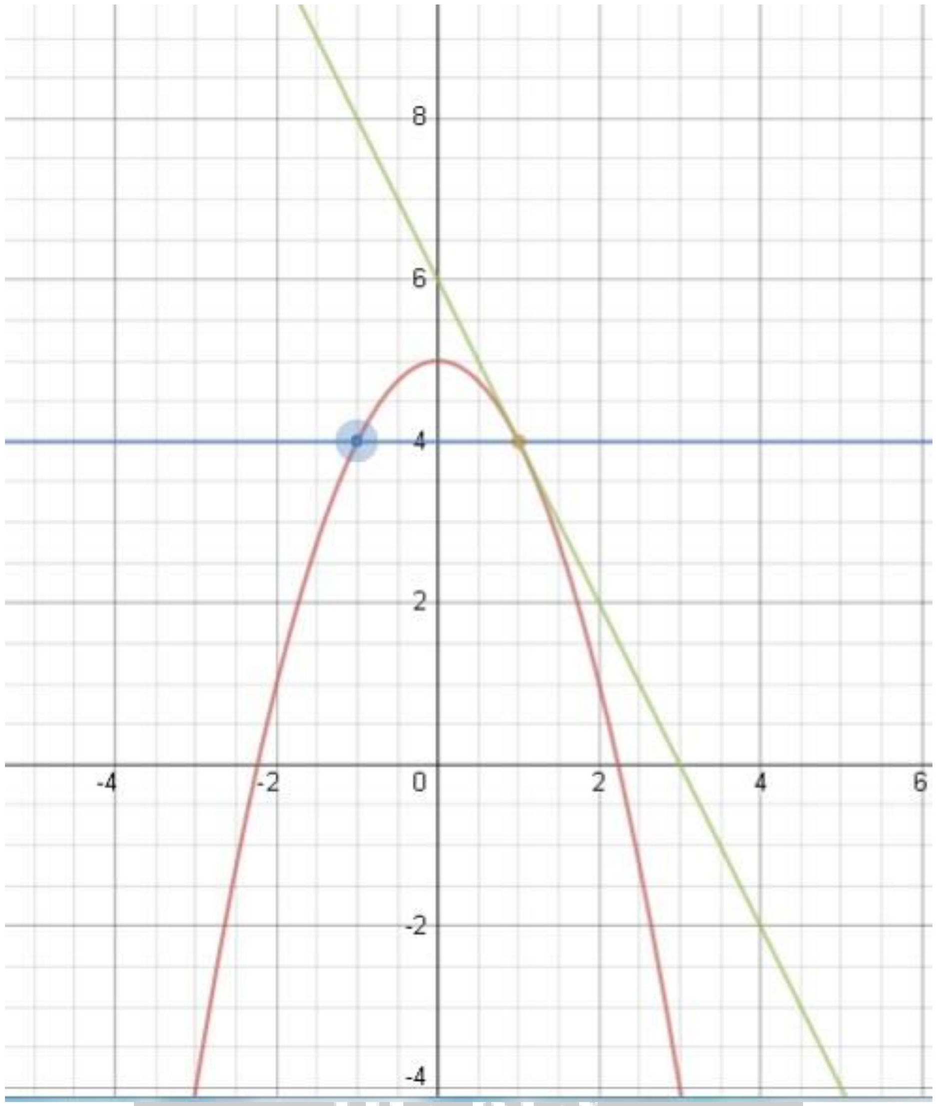
The Tangent Line

In precalculus, we took a look at a secant line. The secant line passes through two points in a curve. On the other hand, the tangent line passes through one point in the curve. The function f and a single point on the graph of f are given so that we can find the equation of the tangent line to the graph of f at the given point.

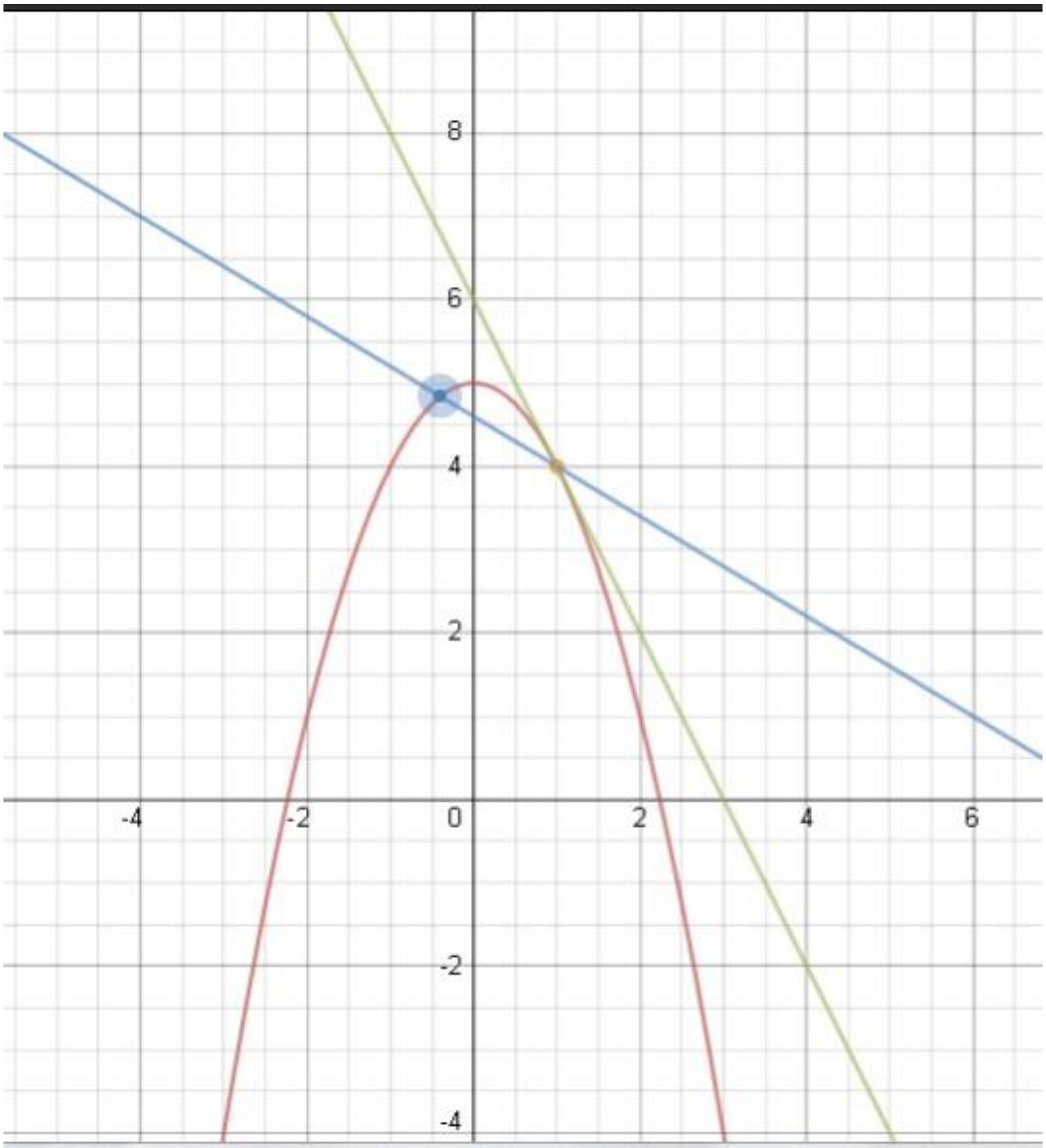
Let A be the point where the tangent line passes through. Let B be the second point that the secant line passes through. (Our secant lines pass through point A and point B)

As point B slowly approaches point A, the value of the slope of the secant line moves closer to the slope of the tangent line:

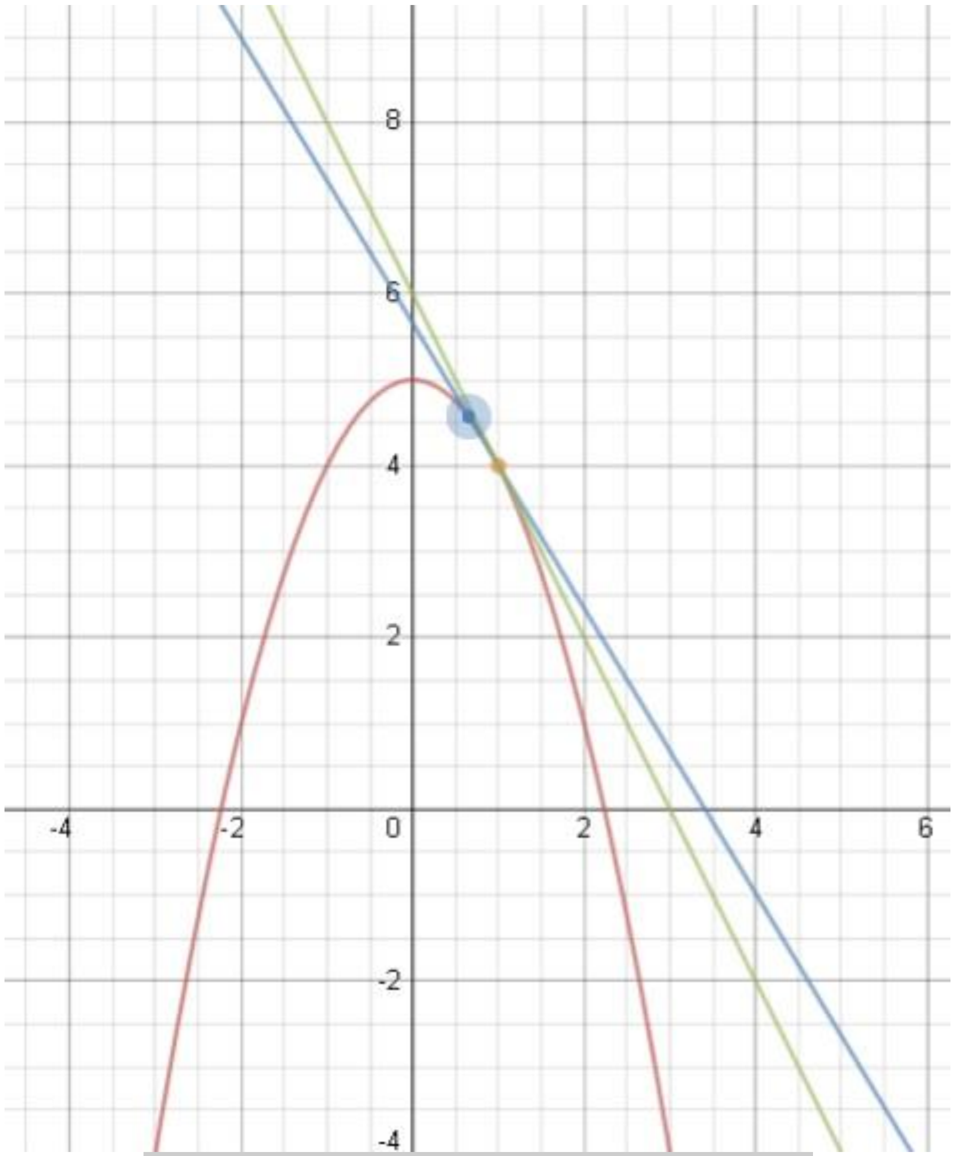
(In the following diagrams, our function f represents the red curve. The **blue line** is the secant line (passing through two points). The **green line** is the tangent line (passing through the single point). As you will from the diagrams, as point B moves closer and closer to point A, the value of the slope of the secant line approaches the value of the slope of the tangent line.)



Point A: The yellow circle
Point B: The blue circle



Point B moving closer to Point A



The slope of the secant line is given as

$$m_{sec} = \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

***f*(c + Δ*x*)** would be the second point chosen as x to form the secant line (x of Point B) plugged into our function f

***f*(c)** would be the value using the given x that the tangent line passes through (x of Point A) plugged into our function f

$\Delta x = x_2 - x_1$
(x_2 would be the second point that we will choose for the secant line to estimate the value of the slope of the tangent line (Point B); x_1 would be the given point that lies on

our curve which is used to find the tangent line (Point A). If it may seem a little confusing, don't worry because we will see this in more detail in our example.

Let's take a look at an example:

Given the function $f(x) = x^3 + 6$, find the line tangent to the graph at $x = 2$.

The first thing we need to do is find a point on the tangent line. From our previous discussion, the point will be A. Using our given information, we find $(x, f(2))$. Plugging in our information, we get (2, 14).

$$f(2) = (2)^3 + 6$$

$$= 14$$

If you remember from our discussion on the secant line, we can choose a second point and estimate the slope of the tangent line. The second point will be our point B (as we had discussed previously). As the point on the secant line moves closer to the point on the tangent line (in our case, it is at (2, 14)), the value of the slope of the secant line will approach the value of the tangent line at that point.

For another point, let's choose $\boldsymbol{x = 3}$.

Using our definitions above:

$f(c + \Delta x) = f(3)$
$f(c) = f(2)$
$\Delta x = 3 - 2$

$$m_{sec} = \frac{f(3) - f(2)}{3 - 2}$$

$$m_{sec} = \frac{[3^3 + 6] - [2^3 + 6]}{1}$$

$$m_{sec} = \frac{39 - 14}{1} = \mathbf{25}$$

Let's now choose points that move closer to our given x:

$\boldsymbol{x = 2.001}$

$$m_{sec} = \frac{f(2.001) - f(2)}{2.001 - 2}$$

$$m_{sec} = \frac{[2.001^3 + 6] - [2^3 + 6]}{.001}$$

$$m_{sec} = \frac{14.012 - 14}{.001} = \mathbf{12.006}$$

Let *x* = 2.0001

$$m_{sec} = \frac{f(2.0001) - f(2)}{2.0001 - 2}$$

$$m_{sec} = \mathbf{12.0006}$$

Our estimate of the slope of the tangent line is **12**.

Later on when we learn the rules for differentiation, we can find the slope of the tangent line at the given point:

$$f(x) = x^3 + 6$$

$$f'(x) = 3x^2$$

$$f'(2) = 3(2)^2$$

$$= 12$$

Don't worry about this explanation for now since we will cover differentiation rules in the next chapter. Right now just follow the explanation when we estimated the slope of the tangent line using the slopes of the secant lines. We will expand on this as we discuss limits.

As observed in our example when point B approached point A, the slope of the secant line moved closer to the slope of the tangent line (where the slope of the secant line reaches its "limit" upon reaching the slope of the tangent line). The slope of the tangent line is then said to be the **limit** of the slope of the secant line. The **limit** is the value that a function approaches as the x approaches a specific value, c. In this case, as the point on the secant line moves closer and closer to the point on the tangent line, the point on the secant line "approaches" the limit.

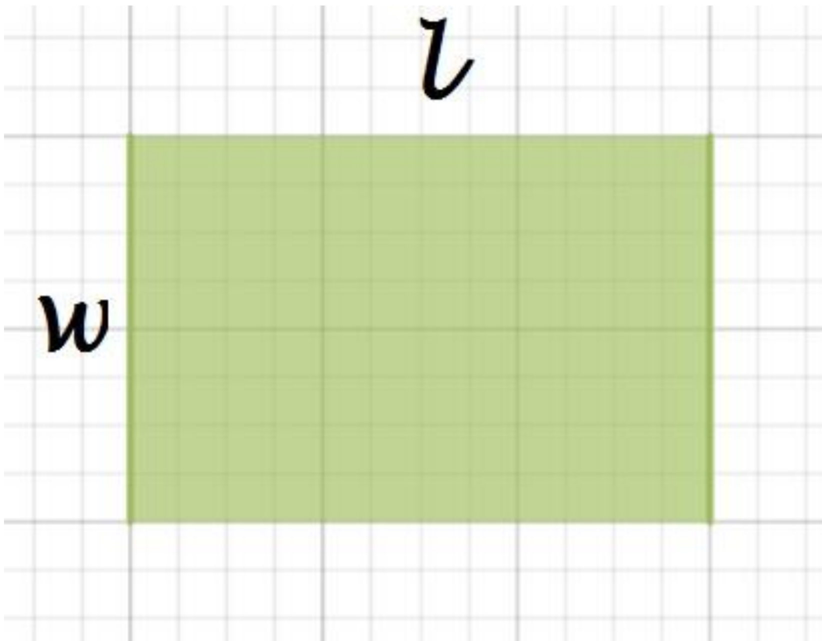
Using our example above, as we moved closer and closer to x = 2 (c = 2), the value of the slope approached the limit, L = 12. Once we discuss this in the section after "The Area Problem," we can denote the example as

$$\lim_{x \to 2} f(x) = 12$$

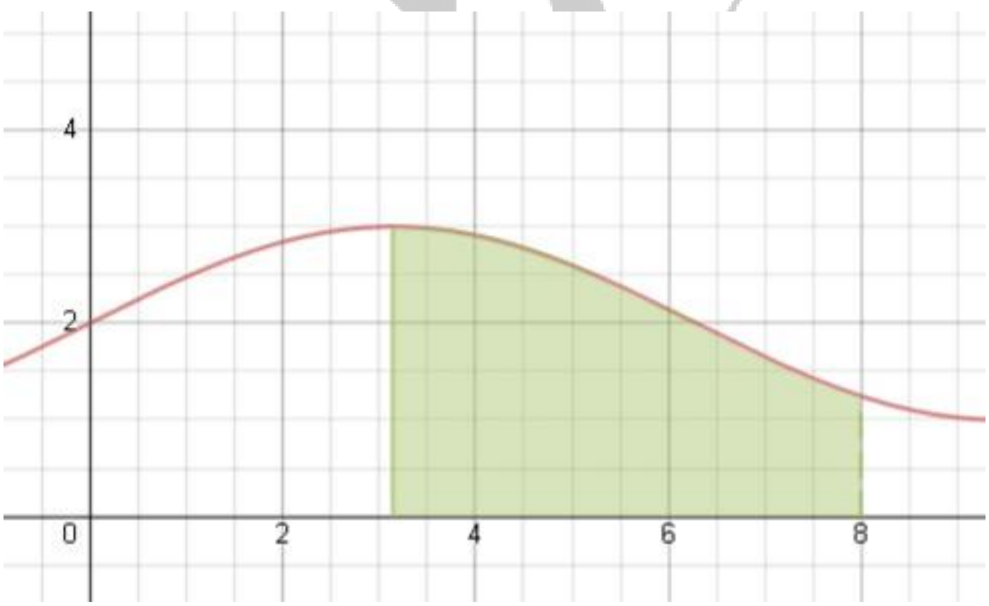
Stated as "**The limit of *f*(*x*) as *x* approaches 2 is 12**"

The Area Problem

In geometry, we found the area of polygons and composite figures. When we found the area of a regular polygon such as a rectangle we noted that $\boldsymbol{A = lw}$.



Now we will examine how we can we find the area of a bounded region under a curve.



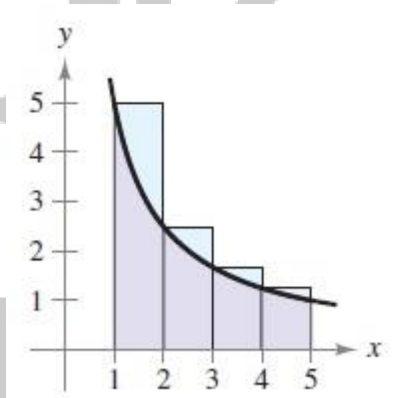
In integral calculus, we will go in depth in regards to finding the area of a bounded region under a curve. However, in this section we will use a method of approximating the area of the bounded region of a given function using rectangles. We will find the limit of the sum of the areas approximated by the rectangles as the number of rectangles approaches infinity (increasing without bound).

Let's take a look at an example from Larson's book (section 2.1, p. 67, #9) and work on it together. I will give a simplified explanation to give you a general overview of the area problem's relevance to calculus. This topic is usually covered at the end of first semester calculus so we will get to it when we discuss integral calculus (the final chapter covered in Calculus I).

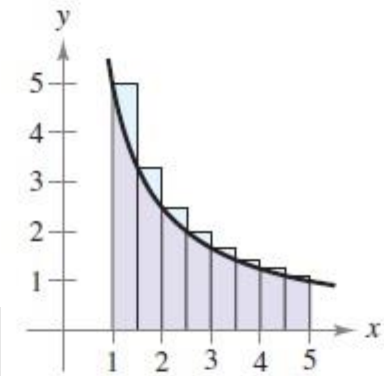
<u>Example</u>:
Use the rectangles in each graph to approximate the area of the region bounded by $y = \frac{5}{x}$, $y = 0$, $x = 1$, and $x = 5$.

We are given two graphs: Graph A uses 4 rectangles to approximate the area and Graph B uses 8 rectangles (double of Graph A).

<u>Graph A</u>



Graph B



In our graphs, the rectangles are circumscribed over the shaded region.

To approximate the area under the curve between $y = \frac{5}{x}$ on the interval [1,5], we divide the interval using rectangles. For Graph A, we divided the interval [1, 5] using four rectangles. For Graph B, we divided the interval [1, 5] using eight rectangles.

$$\boldsymbol{\Delta x = \frac{b - a}{n}}$$

$\boldsymbol{\Delta x}$ denotes the width of the subinterval. In our current cases, the rectangles will be of equal width.

Our interval [a, b] is [1, 5]

n is the number of rectangles (divided subintervals)

We begin with Graph A:

We are given the information to find the width of each subinterval:

$$\Delta x = \frac{b - a}{n}$$

$$= \frac{5 - 1}{4}$$

$$= \frac{4}{4} = 1$$

We use the left endpoint a to estimate the area. For both of our graphs (Graphs A and B), the rectangles include additional area than what is bound under our y. As a result, we will overestimate the area under the curve.

There will be four subintervals each with a width of 1:

[1, 2] [2, 3] [3, 4] [4, 5]

x_i represents the left endpoints (in this example, they will be 1, 2, 3, and 4)

To find the estimated area (A_r is the "area of the rectangles"):

$$A_r \approx \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

For $f(x_i)$, we put the values of the left endpoints into our y [or $f(x)$]. To make it easier, we will write it as $f(x)$. Our $f(x) = \frac{5}{x}$.

As an example, we'll begin with $f(x_0)$.

$$f(x_0) = \frac{5}{x}$$

$$f(1) = \frac{5}{1} = 5$$

Continue to do the same for the rest of the endpoints.

We construct a table to organize our information:

x_i	$x_0 = 1$	$x_1 = 2$	$x_2 = 3$	$x_3 = 4$
$f(x_i)$	5	5/2	5/3	5/4

We sum the values of $f(x_i)$ multiplied by Δx. Since our $\Delta x = 1$, we can just add the values of $f(x_i)$ to get our estimated area.

Before summing the values, we have to find the common denominator. In this case it is 12:

$5 = \frac{60}{12}$	$\frac{5}{2} = \frac{30}{12}$	$\frac{5}{3} = \frac{20}{12}$	$\frac{5}{4} = \frac{15}{12}$

$$A_r = \frac{125}{12}$$

$$\boldsymbol{A_r \approx 10.417}$$

Next we go to Graph B:

$$\Delta x = \frac{b - a}{n}$$

$$= \frac{5 - 1}{8}$$

$$= \frac{4}{8} = \frac{1}{2}$$

Since we doubled the number of rectangles for Graph B, the width of the subintervals is half of the width of the rectangles in Graph A.

Similar to our approach for Graph A, we will use the same steps for Graph B:

There will be eight subintervals each with a width of ½ (or 0.5).

[1, 1.5]	[1.5, 2]	[2, 2.5]	[2.5, 3]	[3, 3.5]	[3.5, 4]	[4, 4.5]	[4.5, 5]

We construct our table [I also put the fraction form of the values of x_i with 0.5 in parentheses to simplify when putting it into $f(x)$]:

$\boldsymbol{x_i}$	$\boldsymbol{x_0 = 1}$	$\boldsymbol{x_1 =}$ **1.5 (3/2)**	$\boldsymbol{x_2 = 2}$	$\boldsymbol{x_3 =}$ **2.5 (5/2)**	$\boldsymbol{x_4 = 3}$	$\boldsymbol{x_5 = 3.5}$ **(7/2)**	$\boldsymbol{x_6 = 4}$	$\boldsymbol{x_7 =}$ **4.5 (9/2)**
$\boldsymbol{f(x_i)}$	5	10/3 or $3.\overline{3}$	5/2 or 2.5	2	5/3 or $1.\overline{6}$	10/7 or $1.4\overline{285714}$	5/4 or 1.25	10/9 or $1.\overline{1}$

$$\boldsymbol{A_r \approx \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x}$$

[f(1)*0.5] + [f(1.5)*0.5] + [f(2)*0.5] + [f(2.5)*0.5] + [f(3)*0.5] + [f(3.5)*0.5] + [f(4)*0.5] + [f(4.5)*0.5]

$$\boldsymbol{A_r \approx 9.145}$$

Now let's compare our values with the area under the curve through the definite integral:

(I'm just going to show you this now so that you can see the approach, but don't worry about it since we will cover it later on in integral calculus. What I want you to look at is the **final answer**.)

$$= \int_1^5 \left(\frac{5}{x}\right) dx$$

$$= 5 \int_1^5 \left(\frac{1}{x}\right) dx$$

$$= 5 \ln|x| \Big|_1^5$$

$$= 5 \ln|5| - 5 \ln|1|$$

$$= 5 \ln 5 - 0$$

$$\approx \mathbf{8.047}$$

As you can see from our example the more rectangles there are, the approximation moves closer to the value of the area (Graph B (**$A_r \approx$ 9.145**) was closer at estimating the area compared to Graph A (**$A_r \approx$ 10.417**)). This is because the increase in the number of rectangles covers more area and decreases the amount of the area missed.

Numerical Approach to Finding Limits

Suppose we are given a function,

$$\boldsymbol{f(x) = \frac{x^2 - 9}{x - 3}, x \neq 3}$$

When we draw the graph of $f(x)$, we can find the value of $f(x)$ for all values of x except $x = 3$. To examine the graph's behavior as x moves closer to 3, we can evaluate x-values close to 3 from both sides: as the values of x approach 3 from the **left** and the values of x approach 3 from the **right**.

When we construct a table of x-values,

x approaches 3 from the left ➔ **x approaches 3 from the right** 🡐

x	2.5	2.9	2.99	2.999	3	3.001	3.01	3.1	3.5
$f(x)$	5.5	5.9	5.99	5.999	?	6.001	6.01	6.1	6.5

When we use limit notation, we can write this as

$$\lim_{x \to 3} f(x) = 6$$

It is read as: "**The limit of *f(x)* as *x* approaches 3 is 6.**"

From this we can see that even though x cannot equal 3, you can choose values that move closer to 3, and $f(x)$ will move closer to 6.

This first example shows us how to estimate the limit numerically. We choose values that move arbitrarily close to the value of x to estimate the value of $f(x)$.

Analytical Approach to Evaluating Limits

In this section we will look at evaluating limits through direct substitution. In this case,

$$\lim_{x \to c} f(x) = f(c)$$

where the limit is precisely at $\boldsymbol{f(c)}$.

Let's take a look at a few Basic Limit Laws:

$$\lim_{x \to c} b = b$$

$$\lim_{x \to c} x = c$$

$$\lim_{x \to c} x^n = c^n$$

where b and c are real numbers and n is a positive integer.

Examples:

$\lim_{x \to c} b = b$ $\lim_{x \to 2} 5 = 5$	$\lim_{x \to c} x = c$ $\lim_{x \to 1} x = 1$	$\lim_{x \to c} x^n = c^n$ $\lim_{x \to 4} x^2 = 4^2 = 16$

In addition to the properties above, there are also other limit rules which we will encounter as we evaluate limits analytically. The limit properties can be applied for two-sided and one-sided limits.

Before we discuss the limit rules and their corresponding examples, let us consider the following as we cover limit properties for scalar multiple, sums, differences, products, quotients, and powers):

Let us assume that the following limits exist where ***b*** and **c** are real numbers and ***n*** is a positive integer:

$$\lim_{x \to c} f(x) = L$$

and

$$\lim_{x \to c} g(x) = K$$

Scalar Multiple:

$$\lim_{x \to c} [bf(x)] = bL$$

$$b\left[\lim_{x \to c} f(x)\right] = bL$$

Example: Given the following information, evaluate the limit.

$$\lim_{x \to c} f(x) = 4$$

Evaluate $\lim_{x \to c} [3f(x)]$.

From our given information, we note that $L = 4$. Using the scalar multiple property rule:

$$3\left[\lim_{x\to c} f(x)\right] = 3[4] = 12$$

$$\lim_{x\to c}[3f(x)] = 12$$

Sum or Difference:

$$\lim_{x\to c}[\boldsymbol{f(x) \pm g(x)}] = \boldsymbol{L \ \pm K}$$

$$\left[\lim_{x\to c} \boldsymbol{f(x)}\right] + \left[\lim_{x\to c} \boldsymbol{g(x)}\right] = \boldsymbol{L + K}$$

$$\left[\lim_{x\to c} \boldsymbol{f(x)}\right] - \left[\lim_{x\to c} \boldsymbol{g(x)}\right] = \boldsymbol{L - K}$$

Example: Given the following information, evaluate the limit.

$$\lim_{x\to c} f(x) = 2$$

$$\lim_{x\to c} g(x) = 5$$

Evaluate $\lim_{x\to c}[\,\boldsymbol{9f(x) - 2g(x)}]$.

In this example we will apply all of the rules that we learned so far. From our given information, **L = 2** and **K = 5**.

$$9\left[\lim_{x\to c} f(x)\right] - 2\left[\lim_{x\to c} g(x)\right] =$$

$$9\,[2] - 2\,[5] = 18 - 10$$

$$= 8$$

$$\lim_{x\to c}[\boldsymbol{9f(x)}] - \lim_{x\to c} \boldsymbol{2g(x) = 8}$$

Product:

$$\lim_{x\to c}[\boldsymbol{f(x)g(x)}] = \boldsymbol{LK}$$

$$\left[\lim_{x\to c} f(x)\right]\left[\lim_{x\to c} g(x)\right] = LK$$

Example: Given the following information, evaluate the limit.

$$\lim_{x\to c} f(x) = 7$$

$$\lim_{x\to c} g(x) = 6$$

Evaluate $\lim_{x\to c}[f(x)g(x)]$.

Similar to our previous examples, we note that $L = 7$ and $K = 6$.

$$\left[\lim_{x\to c} f(x)\right]\left[\lim_{x\to c} g(x)\right] =$$

$$[7][6] = \mathbf{42}$$

$$\lim_{x\to c}[f(x)g(x)] = 42$$

Quotient:

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{L}{K}$$

$$\text{if } \lim_{x\to c} g(x) \neq 0$$

[In other terms, if $K \neq 0$]

$$\frac{\left[\lim_{x\to c} f(x)\right]}{\left[\lim_{x\to c} g(x)\right]}, if\ \lim_{x\to c} g(x) \neq 0$$

Example: Given the following information, evaluate the limit.

$$\lim_{x\to c} f(x) = 10$$

$$\lim_{x\to c} g(x) = 4$$

Evaluate $\lim_{x\to c}\left[\frac{2f(x)}{g(x)}\right]$.

$$\frac{\left[\lim_{x \to c} 2f(x)\right]}{\left[\lim_{x \to c} g(x)\right]} = \frac{2\left[\lim_{x \to c} f(x)\right]}{\left[\lim_{x \to c} g(x)\right]}$$

$$= \frac{2[10]}{4} = \frac{20}{4}$$

$$= 5$$

$$\lim_{x \to c}\left[\frac{2f(x)}{g(x)}\right] = 5$$

Power:

$$\lim_{x \to c}[f(x)]^n = L^n$$

Example: Given the following information, evaluate the limit.

$$\lim_{x \to c} f(x) = 8$$

Evaluate $\lim_{x \to c}[f(x)]^{2/3}$.

Remember from algebra that for **fractional exponents** like the one in our current example:

Top number: The power
(In our current example, it is "2" so we square our given value)

Bottom number: The root
(In our current example, it is "3" so we take the cube root of the square of the value)

(This is important since we will frequently encounter this as we move further in calculus. For example, if we need to evaluate the derivative of $\sqrt{x}$, we will need to first rewrite it in its exponential form as $x^{1/2}$).

$$\lim_{x \to c}[f(x)]^{2/3}$$

$$= \lim_{x \to c}\left[\sqrt[3]{(f(x))^2}\right]$$

$$= \sqrt[3]{(8)^2}$$

$$= \sqrt[3]{64}$$
$$= \mathbf{4}$$

$$\lim_{x \to c} [f(x)]^{2/3} = \mathbf{4}$$

Radical Functions:

Let $\boldsymbol{n}$ represent a **positive** integer:

The limit $\lim_{x \to c} f(x)$ exists for all values of c when n is **odd**.
The limit $\lim_{x \to c} f(x)$ exists if the value of c is positive (i.e. c > 0) when n is **even**.

$$\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$$

Example

Given the following information:

$$\lim_{x \to c} f(x) = 625$$

Evaluate the following limit:

$$\lim_{x \to c} \sqrt[4]{f(x)}$$

$$\lim_{x \to c} \sqrt[4]{f(x)} = \sqrt[4]{625}$$

$$= 5$$

$$\lim_{x \to c} \sqrt[4]{f(x)} = 5$$

In this example, n is even and c is a positive value. Recall from algebra that multiplying a positive number by itself (e.g. 2 x 2 = 4) OR a negative number by itself (e.g. -2 x -2 = 4) will yield a **positive** result. **Thus, a number multiplied by itself cannot have a negative value.** This can be seen through our example when n is even and the value of c is positive (i.e. there is no real number multiplied by itself that can yield a negative value which is why the square root or other even-order roots do not exist in the set of real numbers).

Quick review on imaginary numbers:

Finding the even-order root of a negative number goes into the territory of "imaginary numbers." For example, the square root of -36 is $6i$. Written in mathematical terms: ($\sqrt[2]{-36} = 6i$). As a refresher from algebra, $\sqrt{-\mathbf{1}} = \boldsymbol{i}$.

So for our example, $\sqrt{-36} = \left(\sqrt{-1}\right)\left(\sqrt{36}\right) = (i)(6)$

This gives us $\mathbf{6}\boldsymbol{i}$.

If we think of it as **"for any number when squared will equal 4,"** we can use both examples above (2 x 2 = 4 and -2 x -2 = 4) and state that **<u>all</u>** square roots of 4 will be 2 and -2.

However, the value of *c* **<u>can be negative</u>** when *n* is odd. Let's take a look at an example where this is the case.

<u>Example</u>

Given the following information:

$$\lim_{x \to c} f(x) = -64$$

Evaluate the following limit:

$$\lim_{x \to c} \sqrt[3]{f(x)}$$

Let's review the properties of cubes and cube roots before we work out our example.

When we cube a negative numerical value (let's call it "a"), we get a negative number.

$(-a)^3$ = (-a)(-a)(-a) = negative value

For example,

$(-3)^3$ = (-3)(-3)(-3) = -27

If we find the cube root of -27:

$$\sqrt[3]{-27} = -3$$

Similarly, when we cube a positive numerical value (let's call it "b"), we get a positive number.

$(b)^3 = (b)(b)(b) =$ positive value

For example,

$(3)^3 = (3)(3)(3) = 27$

If we find the cube root of 27:

$$\sqrt[3]{27} = 3$$

These cases can also be extended for the rest of the odd-order roots. As we can see, the cases are valid for **<u>all</u>** values of *c* when *n* is odd.

Now let's return to our example and evaluate our limit:

$$\lim_{x \to c} \left[\sqrt[3]{f(x)}\right]$$

$$= \left[\sqrt[3]{-64}\right]$$

$$= -4$$

$$\lim_{x \to c} \left[\sqrt[3]{f(x)}\right] = -4$$

Polynomial Functions:

Let p(x) represent a polynomial p whose variable is x and let c represent a real number:

$$\lim_{x \to c} p(x) = p(c)$$

For polynomial functions, the direct substitution method is applicable. Let us take a look at an example.

<u>Example</u>

Evaluate the following limit:

$$\lim_{x \to 3} (2x^2 + 7x - 14)$$

Using the limit laws, we rewrite the limit as:

$$= 2\left[\lim_{x\to 3} x^2\right] + 7\left[\lim_{x\to 3} x\right] - \lim_{x\to 3} 14$$

For the first two parts ($2x^2$ and $7x$), we use the scalar multiple rule. For the third part (14), we apply the following basic limit rule: $\lim_{x\to c} b = b$.

Using the direct substitution method, we "substitute" the value of "c" for "x":

$$= 2[3^2] + 7[3] - 14$$

$$= [18 + 21] - 14$$

$$= 39 - 14$$

$$= \mathbf{25}$$

Another approach is to apply our limit law above and note the value of the polynomial p when $x = 3$: $\lim_{x\to c} p(x) = p(c)$:

$$\lim_{x\to 3} p(x) = p(3)$$

$$= 2[3]^2 + 7[3] - 14$$

$$= 25$$

Composite Functions:

Assume that f and g are functions such that $\lim_{x\to c} g(x) = L$ and f is continuous at L,

then

$$\boldsymbol{\lim_{x\to c} f\big(g(x)\big) = f\left(\lim_{x\to c} g(x)\right)}$$

$$= \boldsymbol{f(L)}$$

<u>Example</u>

Evaluate the following limit with the given information:

$$\boldsymbol{f(x) = \sqrt{10 + x}}$$

$$\boldsymbol{g(x) = x^2 + 10x - 18}$$

Evaluate:

$$\lim_{x \to 2} f\big(g(x)\big)$$

Using our rule:

$$\lim_{x \to 2} f\big(g(x)\big)$$

$$= f\left(\lim_{x \to 2} g(x)\right)$$

Step 1: Find $\lim_{x \to 2} g(x)$.

$$\lim_{x \to 2} g(x) = \lim_{x \to 2} x^2 + 10x - 18$$

$$= (2)^2 + 10(2) - 18$$

$$= (4 + 20) - 18$$

$$= 24 - 18$$

$$L = 6$$

Step 2: Find $f(L)$.

Since we learned that

$$\lim_{x \to L} f(x) = f(L)$$

$$f\left(\lim_{x \to c} g(x)\right) = f(L)$$

We now find $f(6)$:

$$\lim_{x \to 6} f(6) = \lim_{x \to 6} \sqrt{10 + x}$$

$$= \sqrt{10 + 6}$$

$$= \sqrt{16}$$

$$= \mathbf{4}$$

$$\lim_{x \to 2} f(g(x)) = 4$$

Rational Functions:

Let r(x) represent a rational function *r* whose variable is *x* and let c represent a real number where q(c) ≠ 0:

$$\lim_{x \to c} r(x) = \lim_{x \to c} \frac{p(x)}{q(x)}$$

$$\lim_{x \to c} r(c) = \lim_{x \to c} \frac{p(c)}{q(c)}, if\ q(c) \neq 0$$

Direct substitution is also applicable to rational functions with nonzero denominators. When the denominator is 0, direct substitution can't be used. When this happens, we will use techniques such as dividing out common factors and rationalizing (multiplying the numerator and denominator by the conjugate) in order to evaluate the limit.

For now we will look at an example where the denominator is not zero.

Example

Evaluate the following limit:

$$\lim_{x \to 2} \frac{9x - 4}{x + 3}$$

When we look at the example, we note that both the numerator and denominator are nonzero. We can apply direct substitution to evaluate the limit since the denominator is zero.

$$= \frac{9\left[\lim_{x \to 2} x\right] - \left[\lim_{x \to 2} 4\right]}{\left[\lim_{x \to 2} x\right] + \left[\lim_{x \to 2} 3\right]}$$

$$= \frac{9[2] - 4}{2 + 3}$$

$$= \frac{14}{5}$$

In the previous section we began to discuss the techniques for dividing out and rationalizing functions. So far when we evaluated the limits of rational functions, our denominators were nonzero. What do we do if the denominator is 0 or we get the indeterminate form 0/0?

When we take the limit of a rational function and the value of the denominator is 0

<u>OR</u>

we take the limit of a rational function and the result is 0/0 (the indeterminate form)

we will use techniques to simplify the functions before taking the limit of the function to check whether the limit is defined.

If the denominator is nonzero, but the numerator is 0 (e.g. 0/4), the limit is 0.

If the numerator is nonzero, but the denominator is 0 (e.g. 4/0), the limit is undefined.

***Let c be a real number and c is in an open interval where *f(x)* = *g(x)* and the two functions agree on all values with the exception of one point.**

If the limit of *g(x)* as *x* approaches c exists, the limit of *f(x)* also exists:

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$$

The first part (marked as *) basically states that the two functions *f* and *g* agree on all points except on one single point. We will visualize this through an example and compare the graph of *f(x)* with the graph of *g(x)*.

<u>**Technique 1: Dividing Out**</u>

Let's take a look at a limit where direct substitution cannot initially be applied.

Evaluate the limit:

$$\lim_{x \to 6} \frac{x^2 - 2x - 24}{x - 6}$$

When we perform direct substitution, the numerator and the denominator give us the indeterminate form 0/0.

For the numerator: $[(6)^2 - 2(6) - 24] = (36 - 12 - 24) = (24 - 24) = 0$

For the denominator: $(6 - 6) = 0$

We will need to check if we can factor the numerator and see if we cancel out with the denominator. When we factor the numerator we get:

$$\lim_{x \to 6} \frac{(\boldsymbol{x} - \mathbf{6})(x + 4)}{(\boldsymbol{x} - \mathbf{6})}$$

$$\lim_{x \to 6} \frac{\cancel{(\boldsymbol{x} - \mathbf{6})}(x + 4)}{\cancel{(\boldsymbol{x} - \mathbf{6})}}$$

After factoring out the numerator, we notice that (x - 6) can be canceled out. When we do that we get:

$$\lim_{\boldsymbol{x} \to \mathbf{6}} (\boldsymbol{x} + \mathbf{4})$$

Now we can take our limit:

$$\lim_{x \to 6} (x + 4)$$

$$= 6 + 4 = 10$$

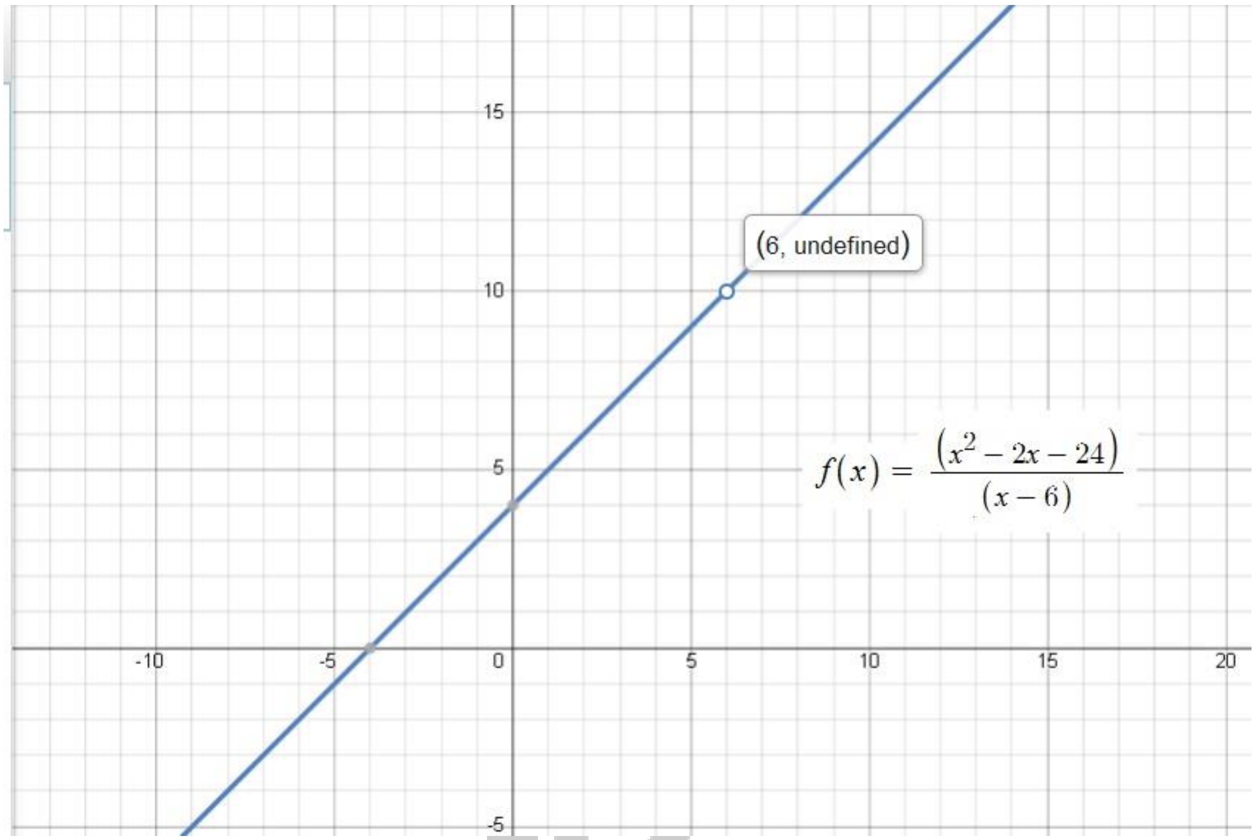
$$\lim_{\boldsymbol{x} \to \mathbf{6}} (\boldsymbol{x} + \mathbf{4}) = \mathbf{10}$$

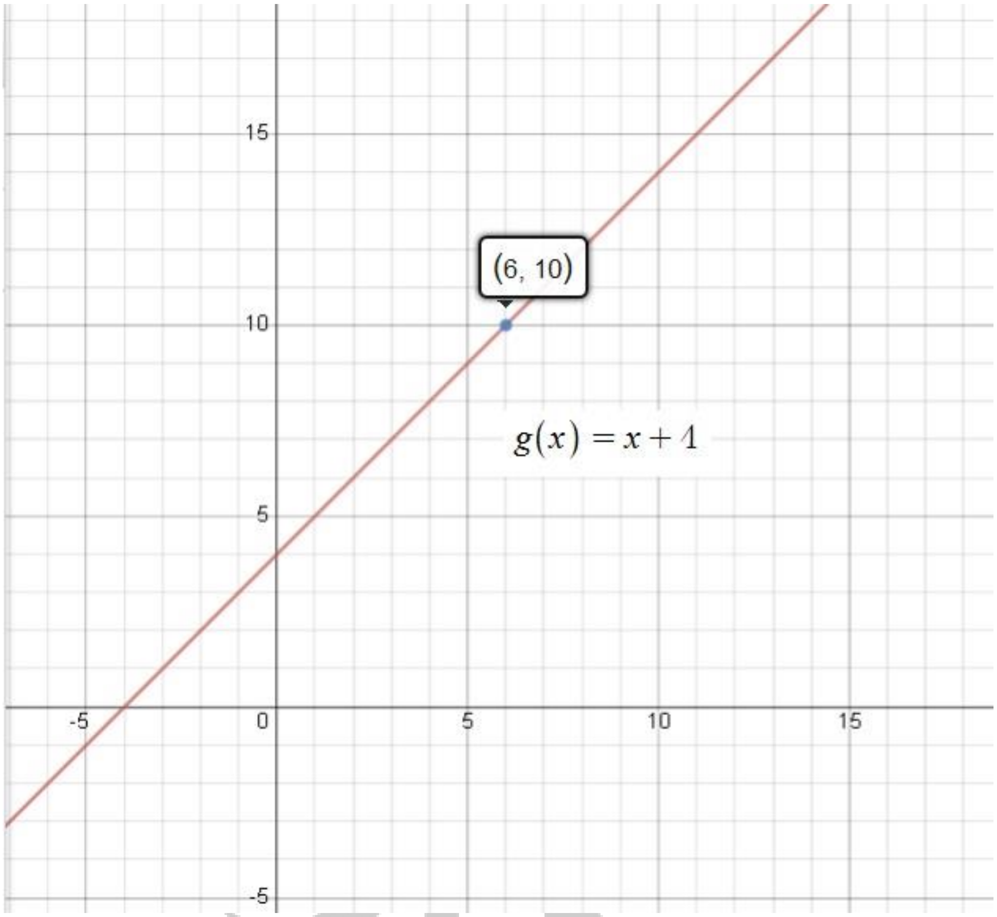
When we compare the two graphs where

$$\lim_{x \to c} f(x) = \lim_{x \to 6} \frac{x^2 - 2x - 24}{x - 6}$$

$$\lim_{x \to c} g(x) = \lim_{x \to 6} x + 4$$

$$\boldsymbol{f}(\boldsymbol{x}) = \frac{\boldsymbol{x}^2 - \mathbf{2}\boldsymbol{x} - \mathbf{24}}{\boldsymbol{x} - \mathbf{6}}$$


(6, undefined)
$f(x) = \frac{(x^2 - 2x - 24)}{(x - 6)}$
15
10
5
0
-5
-10
-5
5
10
15
20



***g(x)* = *x* + 4**

As we can see from the two graphs f(x) and g(x) agree on all points except the single point. Since the limit of *g(x)* as *x* approaches 6 exists, *f(x)* shares the same limit with *g(x)* at *x* = 6.

As we had discussed we conclude that

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = g(c)$$

with *x* approaching *c* (in our example it was 6).

<u>Technique 2: Rationalizing</u>

When evaluating limits, we will encounter numerators or denominators which contain radicals. Taking the limit will result in an indeterminate form. Let's take a look at an example:

$$\lim_{x \to 6} \frac{\sqrt{x+10}-4}{x-6}$$

When we take the limit we will get the indeterminate form:

$$\lim_{x \to 6} \frac{\sqrt{x+10}-4}{x-6}$$

$$= \frac{\sqrt{6+10}-4}{6-6}$$

$$= \frac{4-4}{6-6}$$

$$= 0/0$$

When we encounter these limits, we will need to multiply the numerator and the denominator by the conjugate of the binomial (in our example, the binomial is in the numerator). The conjugate has the opposite sign of the binomial. In our example above our binomial is $\sqrt{x+10}-4$. Therefore we multiply both the numerator and denominator by the binomial's conjugate $\sqrt{x+10}+4$.

When we rationalize the binomial (whether it is in the numerator or the denominator), we remove the radical and simplify. Recall from algebra that performing the FOIL method will take out the "OI" (outside and inside) terms:

The special rule for the difference of squares:

$(a + b)(a - b) = a^2 - b^2$
$(a - b)(a + b) = a^2 - b^2$

Now let's revisit our example and apply what we learned by multiplying by the binomial's conjugate:

$$\lim_{x \to 6} \frac{\sqrt{x+10}-4}{x-6}$$

$$= \lim_{x \to 6} \frac{\sqrt{x+10}-4}{x-6} \frac{\left(\sqrt{x+10}+4\right)}{\left(\sqrt{x+10}+4\right)}$$

$$= \lim_{x \to 6} \frac{(x+10)-16}{(x-6)\left(\sqrt{x+10}+4\right)}$$

$$= \lim_{x \to 6} \frac{x - 6}{(x - 6)\left(\sqrt{x + 10} + 4\right)}$$

The (x - 6) cancels out from the numerator and denominator:

$$= \lim_{x \to 6} \frac{1}{\sqrt{x + 10} + 4}$$

We can now take the limit:

$$= \frac{1}{\sqrt{6 + 10} + 4}$$

$$= \frac{1}{\sqrt{16} + 4} = \frac{1}{4 + 4}$$

$$= \mathbf{\frac{1}{8}}$$

The limit of *f(x)* as *x* approaches 6 is $\frac{1}{8}$.

The Limits of Transcendental Functions

Let *c* be a real number in the domain of the trigonometric function.

$\lim_{x \to c} \sin x = \sin c$	$\lim_{x \to c} \csc x = \csc c$
$\lim_{x \to c} \cos x = \cos c$	$\lim_{x \to c} \sec x = \sec c$
$\lim_{x \to c} \tan x = \tan c$	$\lim_{x \to c} \cot x = \cot c$
$\lim_{x \to c} a^x = a^c, a > 0$	$\lim_{x \to c} \ln x = \ln c$

Similar to our previous examples, we check to see if we can perform direct substitution as in the following examples:

$$\lim_{x \to 0} \cos x = \cos(0) = 1$$

$$\lim_{x \to 0} \frac{\sec x}{e^x + 1} = \frac{\sec(0)}{e^0 + 1} = \frac{1}{1+1} = \frac{1}{2}$$

$$\lim_{x \to e} 5 + \ln x^4$$

$$= \left(\lim_{x \to e} 5\right) + \left(\lim_{x \to e} \ln x^4\right)$$

$$= \left(\lim_{x \to e} 5\right) + \left(\lim_{x \to e} 4 \ln x\right)$$

$$= 5 + 4 \ln e$$

$$= 5 + 4(1)$$

$$= 9$$

However, if the denominator is 0, simplify as best as you can and apply the techniques such as dividing out or rationalizing to evaluate the limit.

Let's take a look at an example:

$$\lim_{x \to 0} \frac{8(e^{2x} - 1)}{e^x - 1}$$

When we perform direct substitution (we can just focus on the denominator in this one), we will get 1- 1 = 0 since $e^0 = 1$. We now have to rewrite and simplify.

The first thing we focus on is the **(e^{2x} - 1)** in the numerator. From our dividing out technique, we learned to find a common factor in order to "divide it out." In this case, we factor out (e^{2x} - 1):

$$= \lim_{x \to 0} \frac{8(e^x + 1)(e^x - 1)}{(e^x - 1)}$$

The **(e^x - 1)** divides out and we can now evaluate the limit:

$$= \lim_{x \to 0} 8(e^x + 1)$$

$$= \left(\lim_{x\to 0} 8\right)\left(\lim_{x\to 0} e^x + 1\right)$$

$$= (8)(e^0 + 1) = (8)(1+1)$$

$$= (8)(2) = \mathbf{16}$$

The Squeeze Theorem

The Squeeze Theorem states that the limit of a function is "sandwiched" or "squeezed" between two other functions. At a specific *x*-value, the limits of all of the functions are the same.

Let's assume

$$\boldsymbol{h(x) \le f(x) \le g(x)}$$

for all x in an open interval containing c (except possibly at c itself) and if

$$\lim_{x\to c} \boldsymbol{h(x) = L = \lim_{x\to c} g(x)}$$

then

$$\lim_{x\to c} \boldsymbol{f(x) = L}$$

exists.

Let's look at an example:

Apply the Squeeze Theorem to find the limit of $f(x)$ as x approaches c with the given information.

Let $\mathbf{c = 0}$

$$\boldsymbol{5 - x^2 \le f(x) \le 5 + x^2}$$

From the Squeeze Theorem, we note that $h(x) = 5 - x^2$ and $g(x) = 5 + x^2$.

Step 1: Determine the limits of $h(x)$ and $g(x)$ as x approaches c.

Let's start with finding the limit of $h(x)$ as x approaches 0.

$$\lim_{x\to c} \boldsymbol{h(x)}$$

$$\lim_{x \to 0} 5 - x^2$$

$$= 5 - (0)^2$$

$$= \mathbf{5}$$

$$\lim_{x \to c} h(x) = L = 5$$

Now let's find the limit of $g(x)$ as x approaches 0.

$$\lim_{x \to c} g(x)$$

$$\lim_{x \to 0} 5 + x^2$$

$$= 5 - (0)^2$$

$$= \mathbf{5}$$

$$\lim_{x \to c} g(x) = L = 5$$

Since

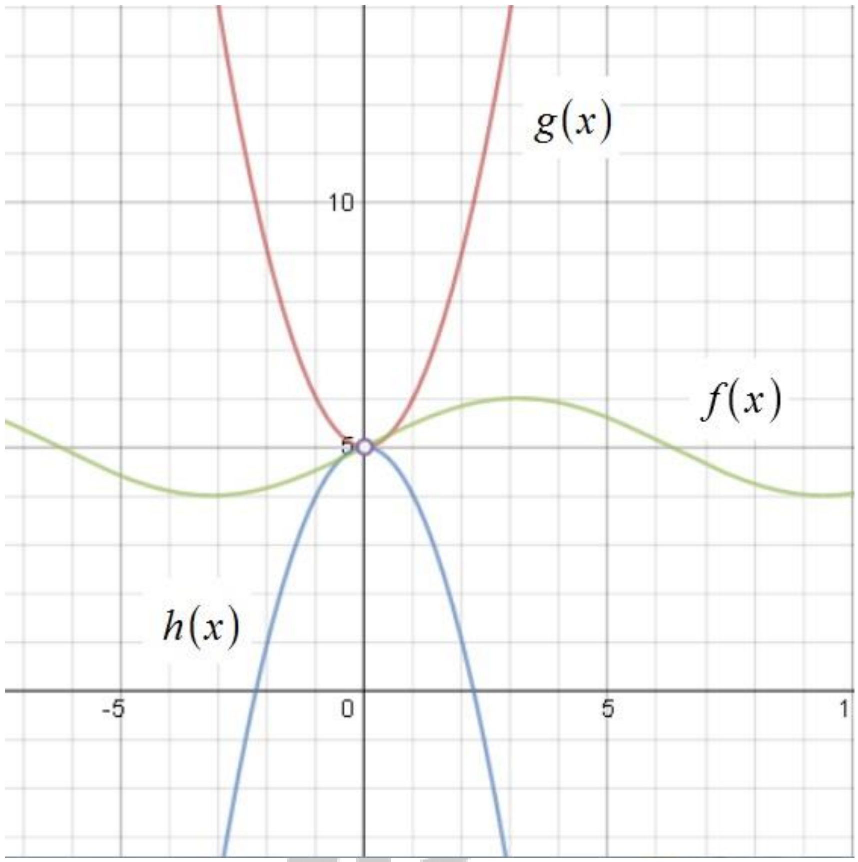
$$\boldsymbol{h(x) \leq f(x) \leq g(x)}$$

and since

$$\lim_{x \to c} h(x) = 5 = \lim_{x \to c} g(x)$$

the limit of $f(x)$ as x approaches 0 is 5.

When we look at the graph for $h(x)$ and $g(x)$, we can see how the limit of f(x) is squeezed between these two functions. Each function also has the same limit (L = 5) at the given x-value, x = 0.



<u>Limits of Trigonometric Functions</u>

Let's take a look at three special limits

$\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$	$\lim\limits_{x\to 0}\dfrac{1-\cos x}{x} = 0$	$\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e$

The Squeeze Theorem is applicable when proving the first two special limits above. For the third special limit, recall from precalculus that for the function, $(x) = (1+x)^{\frac{1}{x}}$ **,** as the values of x get closer to 0, the values of $f(x)$ get closer to e. Earlier in this handout we created a table of values to see the behavior of $f(x)$ as the values of x move closer and closer to c. When we construct the table and use x values that approach 0 from the left and from the right, we see that the values get closer to $e \approx$ **2.71828**.

This is where we get the third special limit where **the limit of $f(x)$ as x approaches 0 is e**.

But for right now you don't need to worry about the specifics. As long as you can understand the main picture that is what's important.

Let’s look at an example:

Determine the limit of

$$\lim_{x \to 0} \frac{\sin^2 x}{x}$$

Looking at the limit of this transcendental function, we can observe that we cannot directly substitute 0 for x. However, we learned from the previous section that we use methods to simplify or rewrite the limit in order for us to evaluate it.

As we look at our three special limits, the first one looks applicable to our current example. To be able to get our special limit we rewrite our original function:

$$\lim_{x \to 0} \frac{\sin^2 x}{x} = \left(\lim_{x \to 0} \frac{\sin x}{x}\right)\left(\lim_{x \to 0} \frac{\sin x}{1}\right)$$

$$= (1)(0)$$

$$= 0$$

A common example that is frequently encountered:

$$\lim_{x \to 0} \frac{\sin 7x}{x}$$

This is the example that will be explained, but many examples will have a similar format such as

$$\lim_{\theta \to 0} \frac{\sin 4\theta}{\theta}$$

$$\lim_{x \to 0} \frac{\sin 8x}{\sin 9x}$$

Let’s return to our example:

$$\lim_{x \to 0} \frac{\sin 7x}{x}$$

Again, we cannot perform direct substitution since we will get the indeterminate form 0/0.

Step 1: Multiply and divide the numerator and denominator by the number located in the numerator.

Our goal is to rewrite the limit so that we can apply the special limits in order to determine the limit of the function.

In our example, we will multiply and divide by 7.

$$\lim_{x\to 0}\frac{(\mathbf{7})(\sin 7x)}{(\mathbf{7})(x)}$$

Step 2: Apply limit laws when rewriting the limit.

For our example, we can use the scalar multiple rule to get:

$$\mathbf{7}\left(\lim_{x\to 0}\frac{\sin 7x}{7x}\right)$$

Step 3: Apply the special limit and evaluate.

After rewriting our limit, we can apply our special limit: $\boldsymbol{\lim_{x\to 0}\frac{\sin x}{x} = 1}$.

Some people prefer to directly evaluate the limit:

$$7\left(\lim_{7x\to 0}\frac{\sin 7x}{7x}\right)$$

$$= (7)(1)$$

$$= 7$$

Other people substitute where we let $7x = \theta$ or let $7x = y$ and then rewrite it as

$$7\left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right)$$

OR

$$7\left(\lim_{y\to 0}\frac{\sin y}{y}\right)$$

The result will be the same:

$$= (7)(1)$$

$$= 7$$

Let's look at another example with the special limit:

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

Determine the limit of the function:

$$\lim_{x \to 0} \frac{\mathbf{1 - \sin x \cot x}}{\boldsymbol{x}}$$

Step 1: Rewrite and simplify the function.

Similar to our previous example, we have to find a way to simplify the function before taking the limit.

The one that comes to mind is the **sin *x* cot *x*** in the numerator. From trigonometry recall that cot x is $\frac{\mathbf{\cos x}}{\mathbf{\sin x}}$. We can then further simplify:

$$\frac{\cancel{\sin x}}{1} \cdot \frac{\cos x}{\cancel{\sin x}}$$

The sin x cancels out and we get **cos *x***.

Step 2: Rewrite the function and determine the limit of the function.

$$\lim_{x \to 0} \frac{1 - \cos x}{x}$$

After rewriting it, we note that this form is the special limit. We then determine the limit:

$$\lim_{x \to 0} \frac{\mathbf{1 - \cos x}}{\boldsymbol{x}} = \mathbf{0}$$

Limits That Fail To (Do Not) Exist

So far our examples looked at limits that exist. However, there are situations when limits fail to exist:

1) When limits have different values from the left hand side and from the right hand side
2) When the limit does not approach a finite number (i.e. the limit of $f(x)$ moves to infinity where it increases or decreases without bound)
3) When the limit of $f(x)$ oscillates, resulting in two different fixed values as x approaches c (i.e. $f(x)$ does not approach a specific value similar to our previous examples in the handout)
4) When x approaches the endpoint of a closed interval (we will talk more about this when we cover one sided limits)

Let's take a look at a few examples that go into more detail regarding these situations where the limits fail to exist.

1. Differing Behaviors From the Left and the Right

For our examples, we have observed that the closer x approached c from both sides, the limit of $f(x)$ was the same value whether x approached c from the left or x approached c from the right.

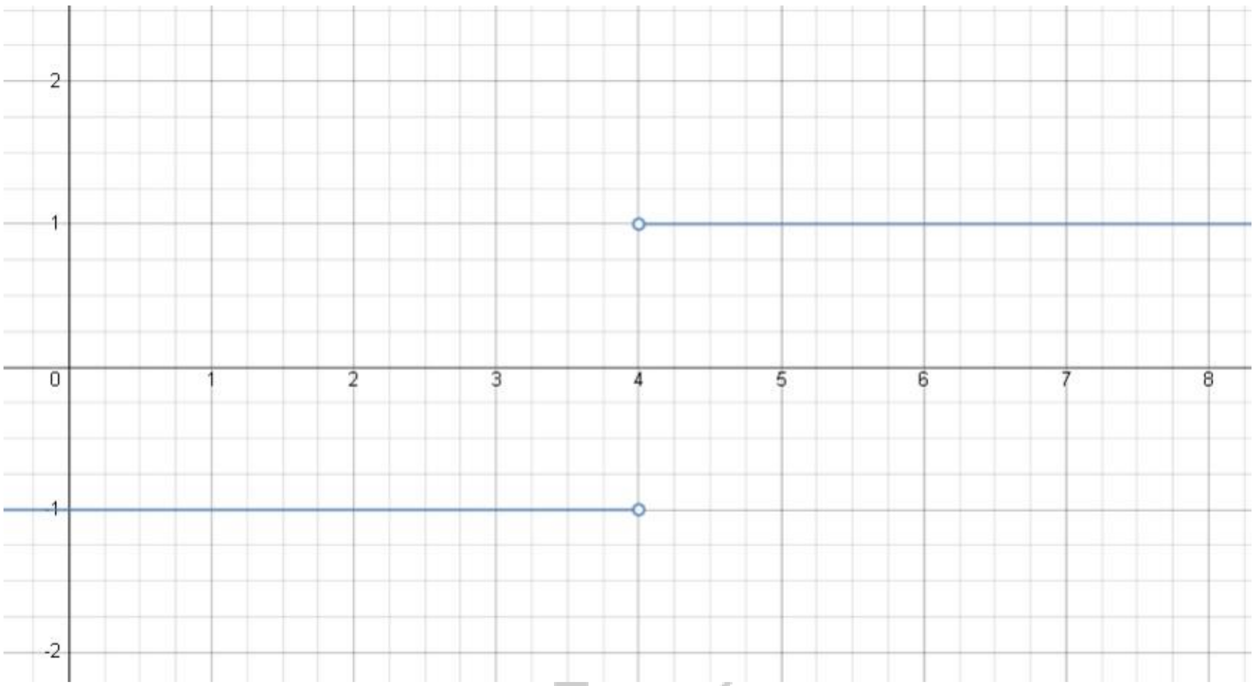
However, when the values differ as x approaches c from the left and from the right (in the case where $f(x)$ approaches two distinct values), the limit of the function does not exist.

Let's take a look at an example:

Determine the limit of the function using the given information. If it does not exist, please explain why.

$$\lim_{x \to 4} \frac{|x-4|}{x-4}$$

When we graph the function:



As we can see from the graph, the value of $f(x)$ is different when x approaches 4 from the left and when x approaches 4 from the right.

Regardless of how close x moves to 4, the values of $f(x)$ are negative as it is approached from the left and positive when approached from the right.

Recall from algebra the definition of absolute value:

$$|\boldsymbol{x}| = \begin{cases} \boldsymbol{x}, \boldsymbol{if}\ \boldsymbol{x} \geq \boldsymbol{0} \\ -\boldsymbol{x}, \boldsymbol{if}\ \boldsymbol{x} < \boldsymbol{0} \end{cases}$$

The value of the absolute value is always zero or positive. We can observe this in our graph:

$$\frac{|x-4|}{x-4} = \begin{cases} 1, if\ x \geq 4 \\ -1, if\ x < 4 \end{cases}$$

Since the behaviors are different from the left and from the right side, the limit does not exist.

2. Infinite/Unbounded Behavior

Let's take a look at an example where the function increases or decreases without bound.

Examine the behavior of

$$\lim_{x \to 3} \frac{1}{x^2 - 9}$$

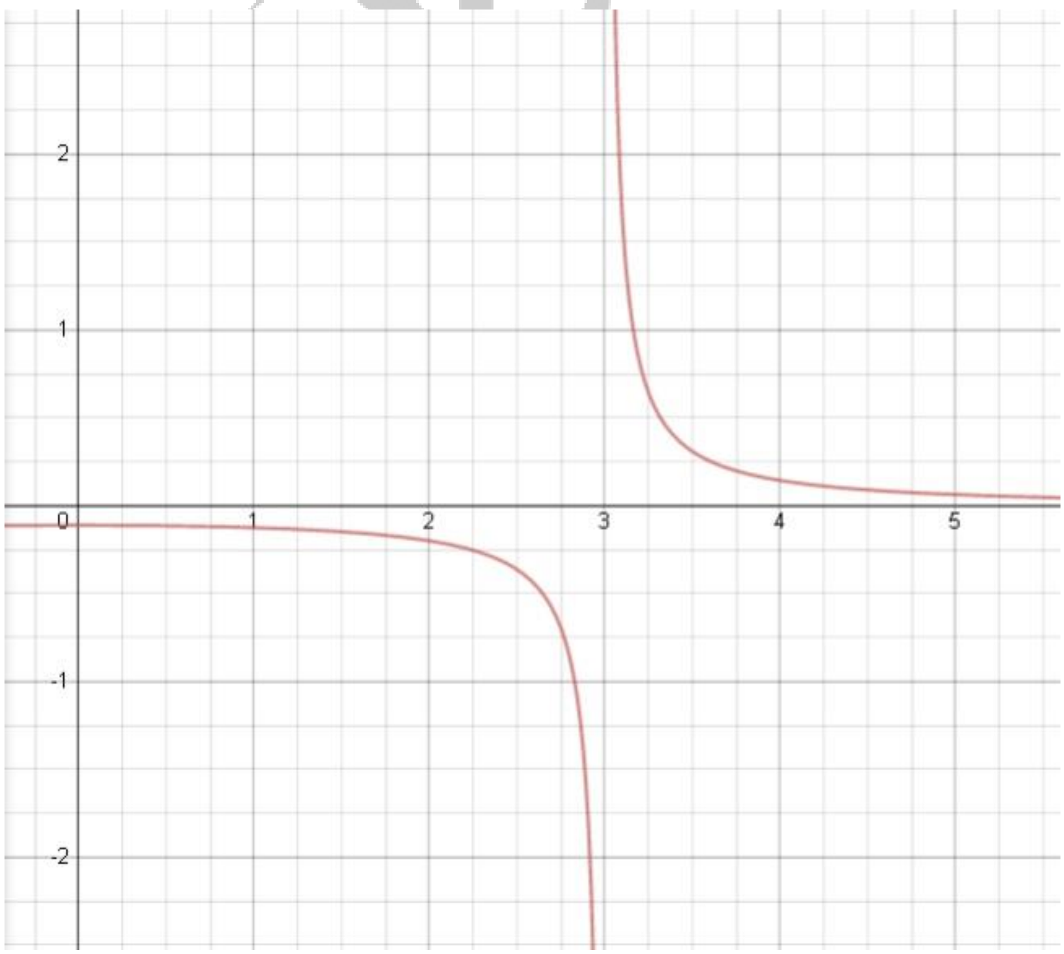
A good way to look at this example is to construct a table of values and/or graph our function to see the behavior as x approaches c from the left and from the right.

x approaches 3 from the left ➔ **x approaches 3 from the right** 🡐

x	2.9	2.99	2.999	2.9999	3	3.0001	3.001	3.01	3.1
f(x)	-1.69	-16.69	-166.69	-1.666.69	?	1666.63	166.63	16.63	1.63

From our table we can see what happens to the function as *x* approaches *c* from the left and from the right. As *x* approaches 3 from the left, the values move to -∞. As *x* approaches 3 from the right, the values move to +∞.

When we graph the function, we can visualize this case:



Since x is not approaching a real number (i.e. where L is a real number), the limit does not exist/fails to exist. In our case, $\lim\limits_{x \to 3} \frac{1}{x^2 - 9}$ **does not exist**.

3. Oscillations

A common example of oscillating behavior can be seen in the graphs of $f(x) = \cos\frac{1}{x}$ and $f(x) = \sin\frac{1}{x}$. When x approaches 0, our functions oscillate between 1 and -1 (i.e. $f(x) = 1$ or -1).

Examine the behavior of

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$

Again, let's construct our table of values:

***x* approaches 0**
➔

x	$\frac{2}{\pi}$	$\frac{2}{3\pi}$	$\frac{2}{5\pi}$	$\frac{2}{7\pi}$	$\frac{2}{9\pi}$	$\frac{2}{11\pi}$	$\frac{2}{13\pi}$	$\frac{2}{15\pi}$	$\frac{2}{17\pi}$
f(x)	1	-1	1	-1	1	-1	1	-1	1

Recall from trigonometry that $\sin\frac{\pi}{2} = 1$. Since we are dividing, we can examine this oscillating behavior using the pattern:

$$x = \frac{2}{z(\pi)}$$

where (z = an odd integer greater than 0 for our example)
note that if we let z = an even integer, we will get 0

Let's do the first two together:

$$f(x) = \sin\left(\frac{1}{x}\right)$$

Let $x = \frac{2}{(1)\pi}$:

$$= \sin\left(\frac{1}{\left(\frac{2}{\pi}\right)}\right)$$

***Note:**

$$\frac{1}{1} \div \frac{2}{\pi} = \frac{1}{1} \cdot \frac{\pi}{2} = \boldsymbol{\frac{\pi}{2}}$$

Continuing on:

$$= \sin\left(\frac{\pi}{2}\right)$$

$$= \mathbf{1}$$

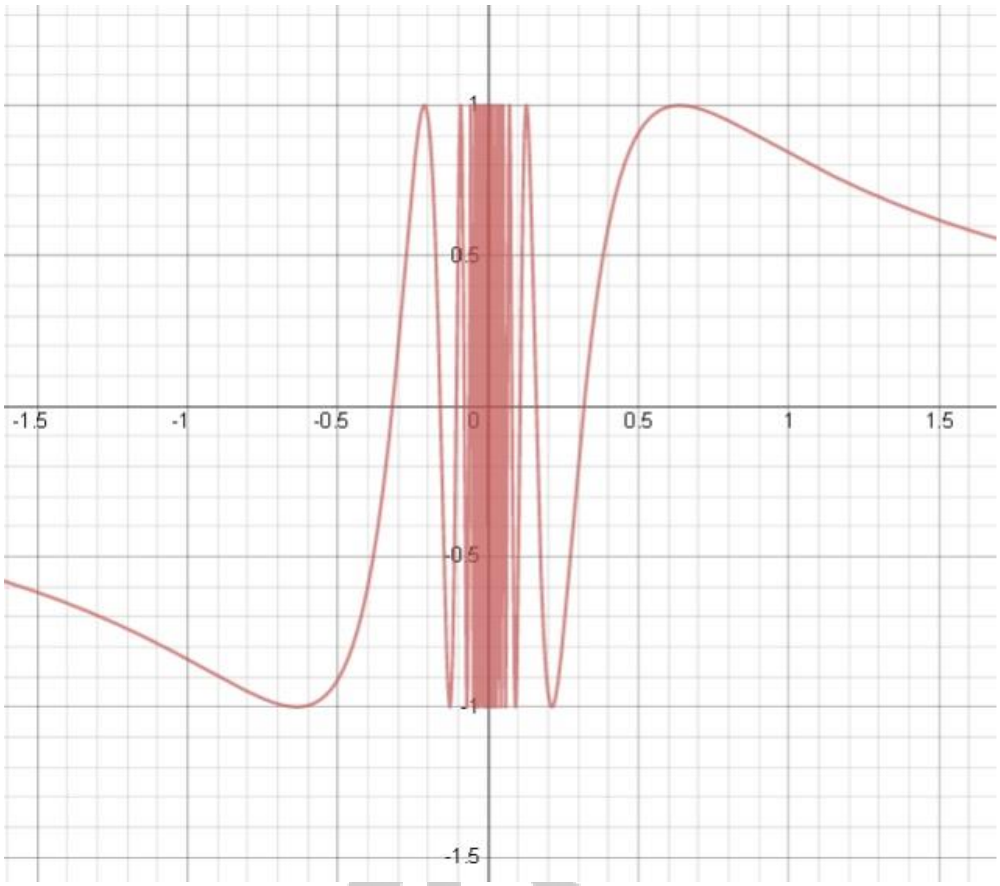
Next we go to the next value $\boldsymbol{x = \frac{2}{3\pi}}$:

$$= \sin\left(\frac{1}{\left(\frac{2}{3\pi}\right)}\right)$$

$$= \sin\left(\frac{3\pi}{2}\right)$$

$$= \mathbf{-1}$$

Let's now graph our function:



From our graph we can visualize the behavior of the function as *x* approaches *c*. The limit of the function does not exist since the value of *f*(*x*) oscillates between 1 and -1.

4. *x* approaches the endpoint of a closed interval

A good example of this is when we try to find a limit of a radical. Recall that the limit of a function exists if *f*(*x*) approaches the same value from the left and from the right. Let's take a look at our earlier table:

$$f(x) = \frac{x^2 - 9}{x - 3}, x \neq 3$$

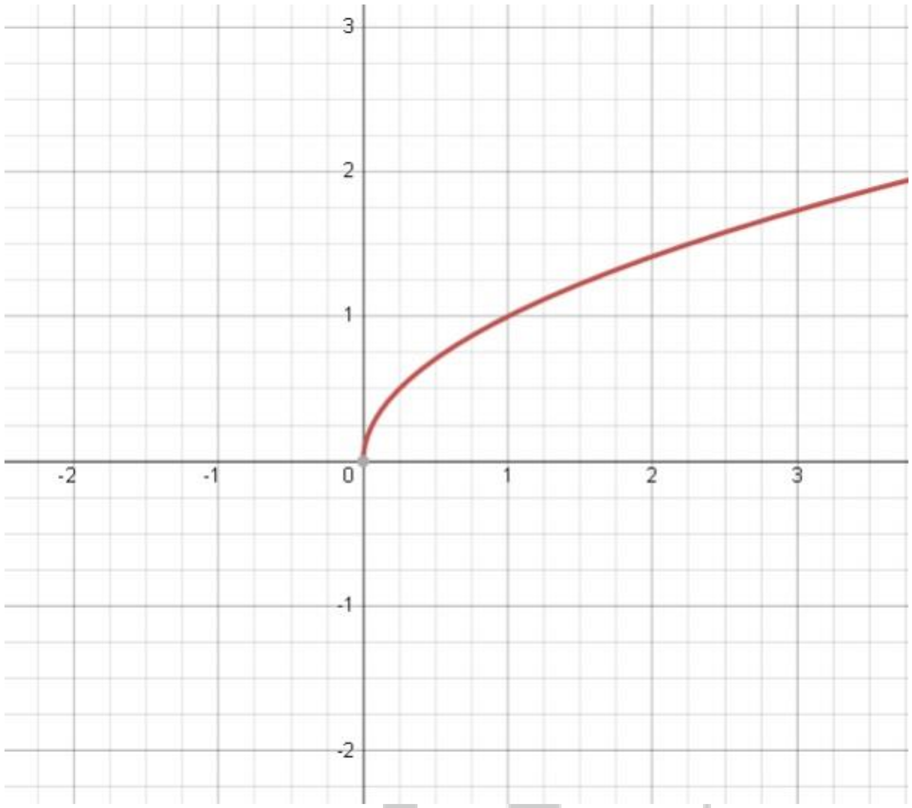
x approaches 3 from the left ➔ **x approaches 3 from the right** 🡐

x	2.5	2.9	2.99	2.999	3	3.001	3.01	3.1	3.5
f(x)	5.5	5.9	5.99	5.999	?	6.001	6.01	6.1	6.5

In our table, *f*(*x*) approaches 6 as *x* moves closer to 3 in both directions.

This is not the case for $\boldsymbol{f(x) = \sqrt{x}}$. Since the value inside the radical cannot be negative, $f(x)$ cannot approach 0 from the left.

When we graph the function $\boldsymbol{f(x) = \sqrt{x}}$:



When we try to find $\lim_{x \to 0} \sqrt{x}$, we can only use positive values where x approaches 0 from the right. **Since we cannot approach 0 from the left, the limit cannot exist.**

So far, we looked at the different ways of looking at and evaluating limits: **numerically** (by creating a table of values), **graphically** (by creating and looking at the graphs), and **analytically** (by using algebra [and later on, calculus] to determine the limit).

In the next few sections, we will look at continuity and finally finish up with infinite limits and the limit definition of the derivative.

<u>Limits and Continuity</u>

A function is continuous at $x = c$ when the following conditions are observed:

1) $f(c)$ is defined

2) $\lim_{x \to c} f(x)$ **exists**
3) $\lim_{x \to c} f(x) = f(c)$

When one of these conditions are not met (i.e. are not true), the continuity of the function is broken.

Functions that are continuous on the entire real number line (-∞, ∞) are **everywhere continuous**.

There are two types of discontinuities which we will take a look at:

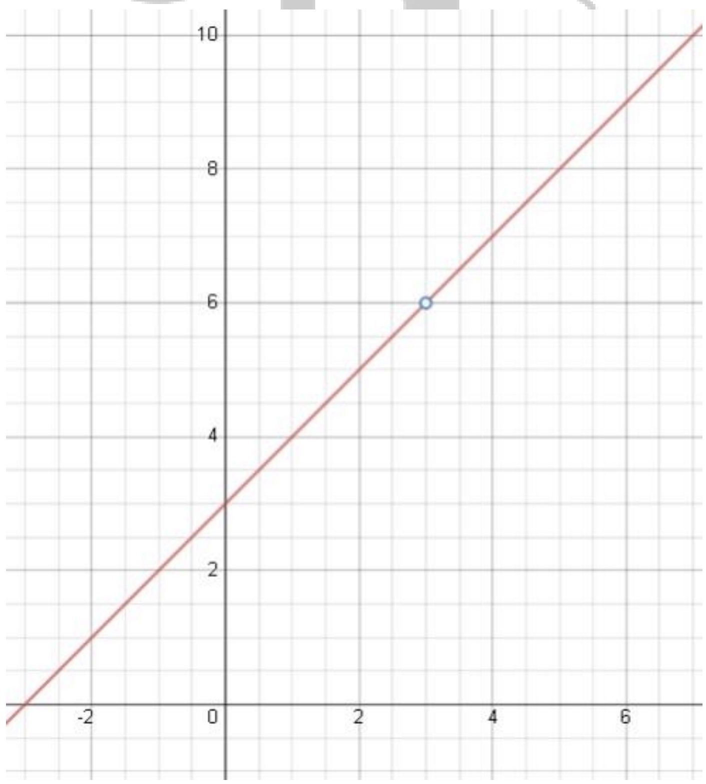
- **Removal discontinuity**
- **Nonremovable discontinuity**

In a **removable discontinuity**, the function can be made continuous if $f(c)$ is defined or redefined.

When we look at the continuity of the function

$$f(x) = \frac{x^2 - 9}{x - 3}$$

and its corresponding graph



The domain of our function is all real numbers except at $x = 3$. We can see from the graph that there is a removable discontinuity at $x = 3$. The function is continuous for all x-values within the specified domain (all real numbers except at $x = 3$).

We can make this function continuous by redefining f(c) (in our case, $f(3)$). If you remember our discussion of Technique 1: Dividing Out, we can redefine $f(c)$ as follows:

$$\lim_{x\to 3}\frac{x^2-9}{x-3}$$

$$f(x)=\frac{x^2-9}{x-3}$$

$$=\frac{(x+3)(x-3)}{(x-3)}$$

$$=\frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}}$$

$$= x + 3$$

When we apply the theorem that we discussed previously:

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x)$$

$$\lim_{x\to 3}\frac{x^2-9}{x-3} = \lim_{x\to 3} x+3$$

$$= \mathbf{6}$$

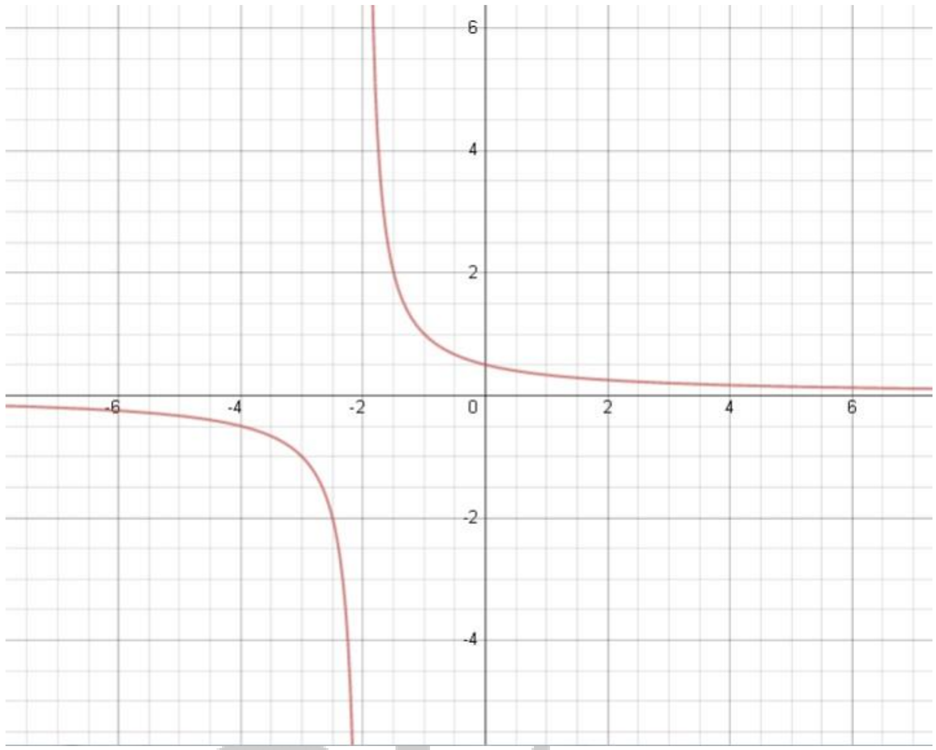
Now that we defined $g(3) = 6$, the function is continuous for all real numbers.

However in a **nonremovable discontinuity,** the function cannot be redefined like in a removable discontinuity.

For example let's take a look at the function

$$f(x)=\frac{1}{x+2}$$

and its corresponding graph



We can observe that at x = -2 there is a nonremovable discontinuity. It is impossible to define f(-2) and make the function continuous at x = -2.

One-Sided Limits

In the earlier section when we discussed limits that failed to exist, we looked at **#4** where x approached the endpoint of a closed interval (as in the case of $\lim_{x \to 0} \sqrt{x}$).

When we begin to look at continuity on a closed interval, we can examine the behavior of functions as x approaches c from the left or from the right. As in the case of $\lim_{x \to 0} \sqrt{x}$, we determined the limit of $f(x) = \sqrt{x}$ as x approached 0 from the right:

if we let n = an even integer,

$$\lim_{x \to 0^+} \sqrt[n]{x}$$

Left-hand limit:

$$\lim_{x \to c^-} f(x) = L$$

where x approaches c from the left

(the x values are less than c: if x approaches 2 from the left, the x values would be examples such as 1.5, 1.9, or 1.99)

<u>Right-hand limit:</u>

$$\lim_{x \to c^+} f(x) = L$$

where x approaches c from the right

(the x values are greater than c: if x approaches 2 from the right, the x values would be examples such as 2.01, 2.1, or 2.5)

We had discussed previously that a two-sided limit does not exist if the behavior from the left and the right exhibit different behavior (i.e. a limit's value from the left is not equal to the limit's value from the right).

The limit of a function exists when

$$\lim_{x \to c^-} f(x) = L \;\; and \;\; \lim_{x \to c^+} f(x) = L$$

where c and L are real numbers.

Using this information, we will extend the definition to continuity of a function on a closed interval.

<u>Continuity on a Closed Interval</u>

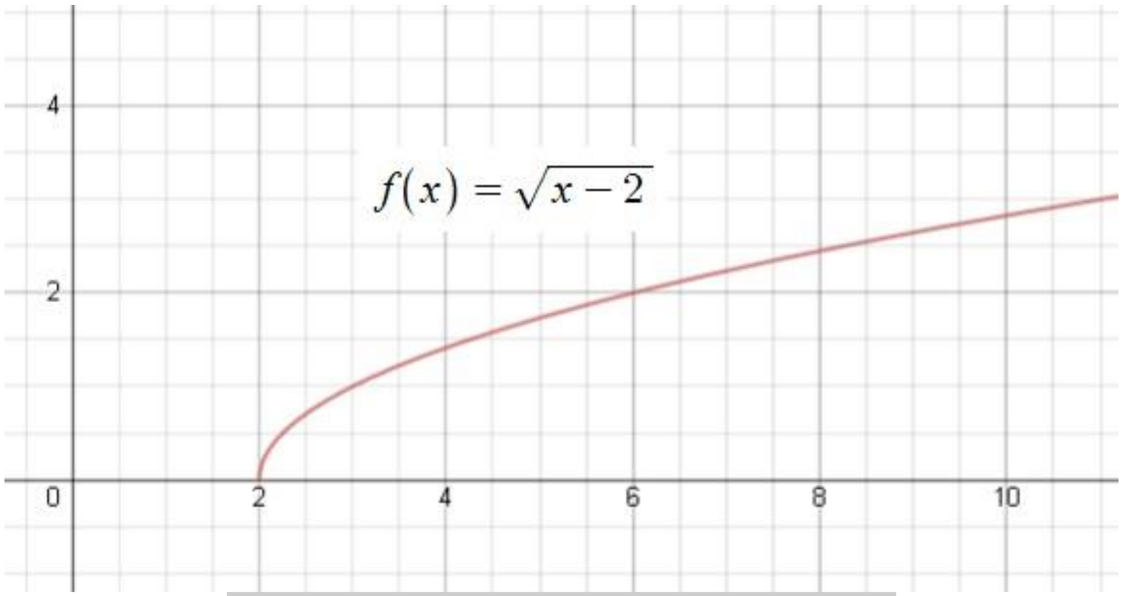
When we defined continuity, we examined continuity on an open interval. However, how do we approach continuity from a closed interval? A function is continuous on a closed interval if

- it is continuous within the interval

  and

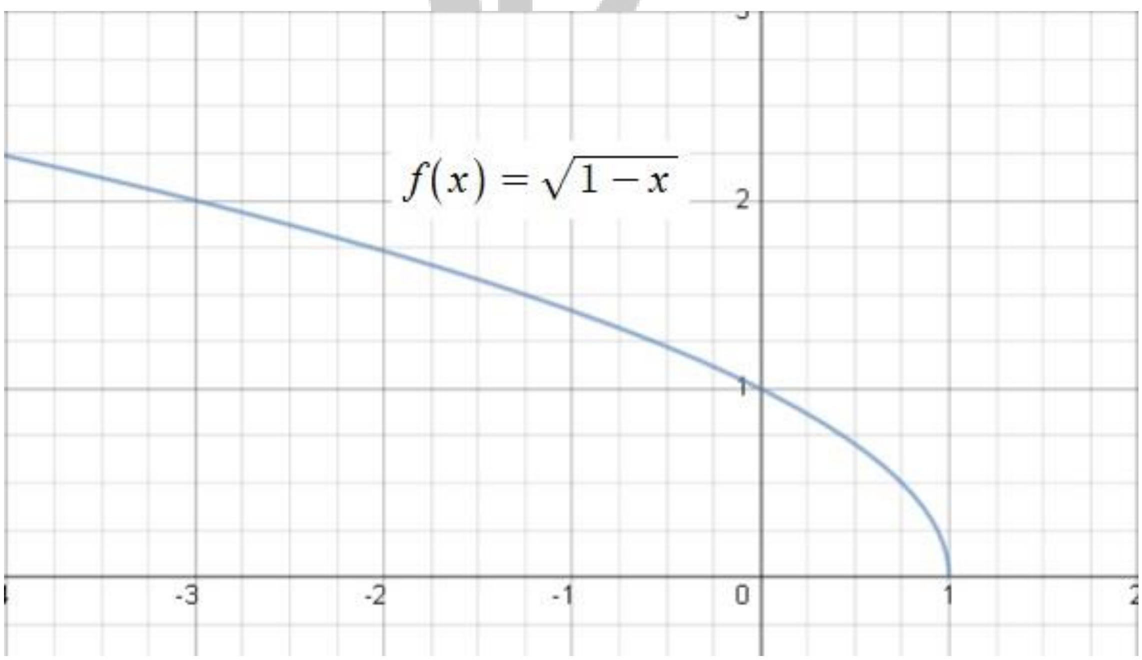
- each endpoint displays one-sided continuity

An example that we looked at was a radical function. Let's examine one-sided continuity of a radical function:



$$\lim_{x \to 2^+} \sqrt{x-2} = f(2)$$

The function is continuous from the <u>right</u> at 2.



$$\lim_{x \to 1^-} \sqrt{1-x} = f(1)$$

The function is continuous from the <u>left</u> at 1.

When we discuss the continuity of the function of a closed interval:

A function is continuous on the closed interval [a, b] if

- **the function is continuous on the open interval (a, b)**
- **the function is continuous from the right at x = a if**

  $$\lim_{x \to a^+} f(x) = f(a)$$

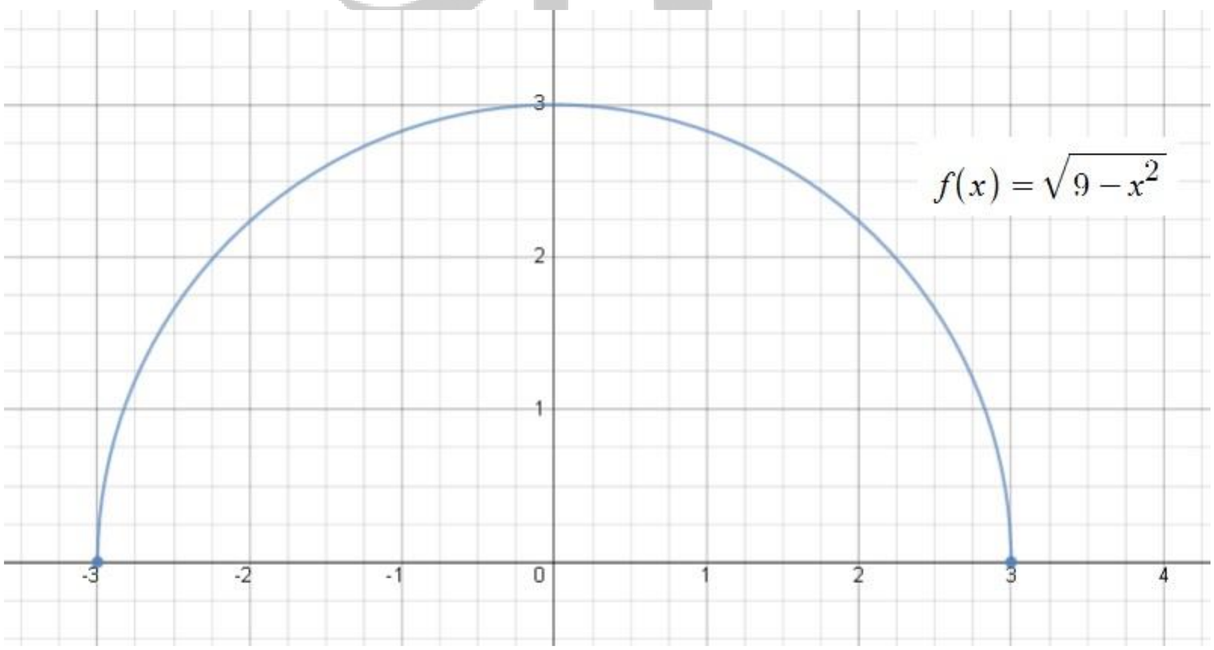
- **the function is continuous from the left at x = b if,**

  $$\lim_{x \to b^-} f(x) = f(b)$$

Let's take a look at an example.

Examine and describe the continuity of $f(x) = \sqrt{9 - x^2}$.

Let's begin by graphing our function:



We can observe that the domain of the function is the closed interval [-3, 3].

When we examine continuity from the left:

$\boldsymbol{f(3)}$ =

$$\lim_{x \to 3^-} \sqrt{9 - x^2}$$

= 0

When we examine continuity from the right:

$\boldsymbol{f(-3)}$ =

$$\lim_{x \to -3^+} \sqrt{9 - x^2}$$

= 0

When we look at our definition of continuity on a closed interval and apply it to our example:

The function *f* is continuous on the open interval (a, b)
Yes, this is true. The function $\sqrt{9 - x^2}$ is continuous on the open interval (-3, 3). If you recall, a function is continuous on an open interval if it is continuous at every x-value in the interval. We can see that from our graph.

The function *f* is continuous from the right at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$
Yes this is true. The endpoint at $x = -3$ exhibited one-sided continuity. The function is continuous from the right at -3.

The function *f* is continuous from the left at $x = b$ if $\lim_{x \to b^-} f(x) = f(b)$
Yes this is true. The endpoint at $x = 3$ exhibited one-sided continuity. The function is continuous from the left at 3.

Properties of Continuity

If f and g are continuous at $x = c$, these functions are also continuous at c:

Scalar multiple:
(let b be a real number)
$\boldsymbol{bf}$

Sum or Difference:
$\boldsymbol{f \pm g}$

Product:
$\boldsymbol{fg}$

Quotient:

$$\frac{f}{g}, if\ g(c) \neq 0$$

These elementary functions are continuous at every point in their domain:

Polynomial:

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

Rational:

$$r(x) = \frac{p(x)}{q(x)}, q(x) \neq 0$$

Radical:

$$f(x) = \sqrt[n]{x}$$

Trigonometric:

$\sin x$, $\cos x$, $\tan x$

$\csc x$, $\sec x$, $\cot x$

Exponential:

$$f(x) = a^x$$
$$f(x) = e^x$$

Logarithmic:

$$f(x) = \ln x$$

<u>Composite Functions and Continuity:</u>

We took a look at the limit of composite functions earlier and from there we will look at the continuity of a composite function.

If g is continuous at c and f is continuous at $g(c)$, then the composite function given by $(f \circ g)(x) = f(g(x))$ is continuous at c.

A few examples of composite functions include:

$$g(x) = \cos 2x$$

$$f(x) = \sqrt{36 - x^2}$$

Let’s take a look at our second composite function example above ($f(x) = \sqrt{36 - x^2}$) and describe its continuity.

Let

$$f(x) = \sqrt{x}$$

$$g(x) = 36 - x^2$$

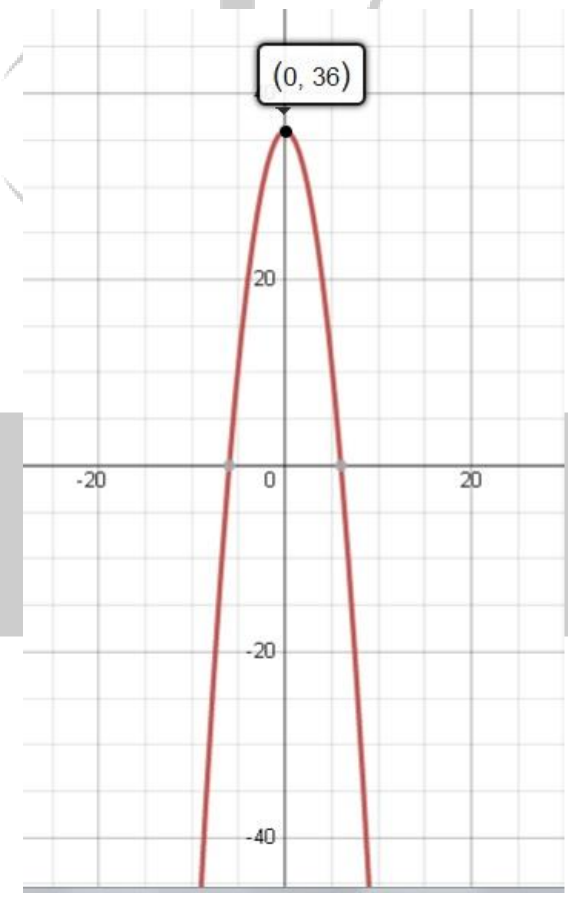
When we simplify $(f \circ g)(x) = f(g(x))$:

$$f(36 - x^2)$$

$$= \sqrt{36 - x^2}$$

The domain in our composite function ($f \circ g$) includes all x in the domain of g that is within the domain of f.

When we look at the domain of g: [$g(x) = 36 - x^2$]

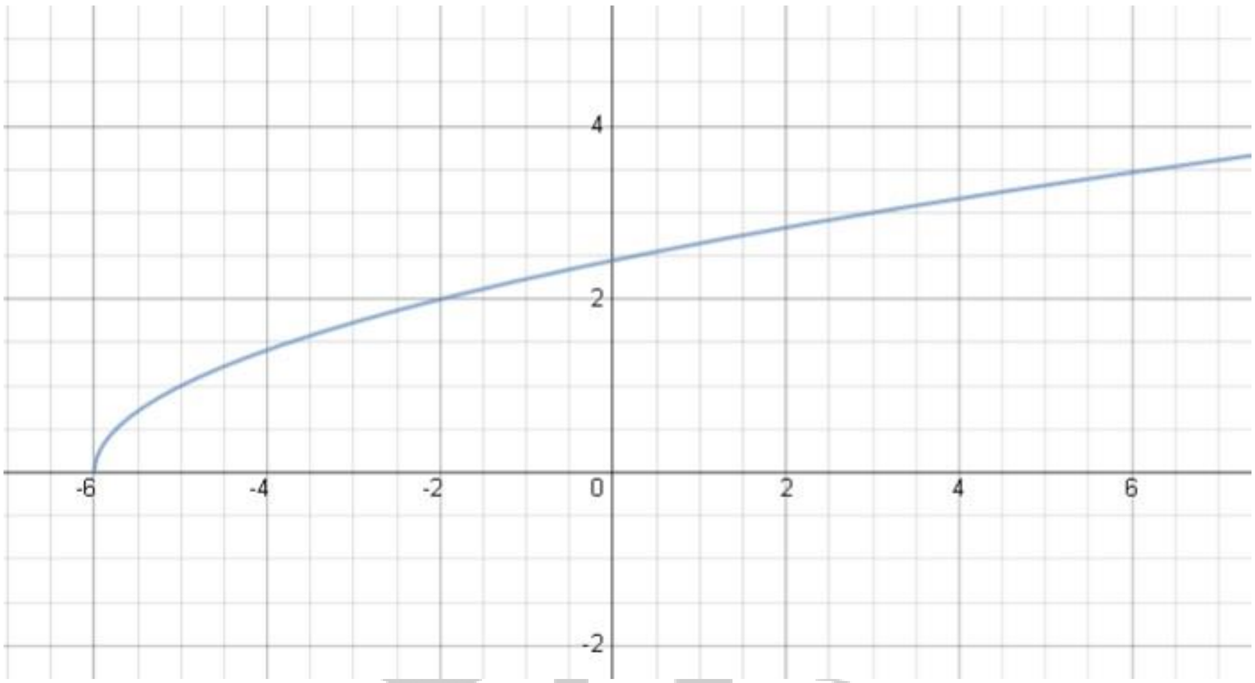


We can see from the graph that there is no restriction on the domain. The domain is (-∞, ∞).

Now we need to find the domain of g that is within the domain of f:

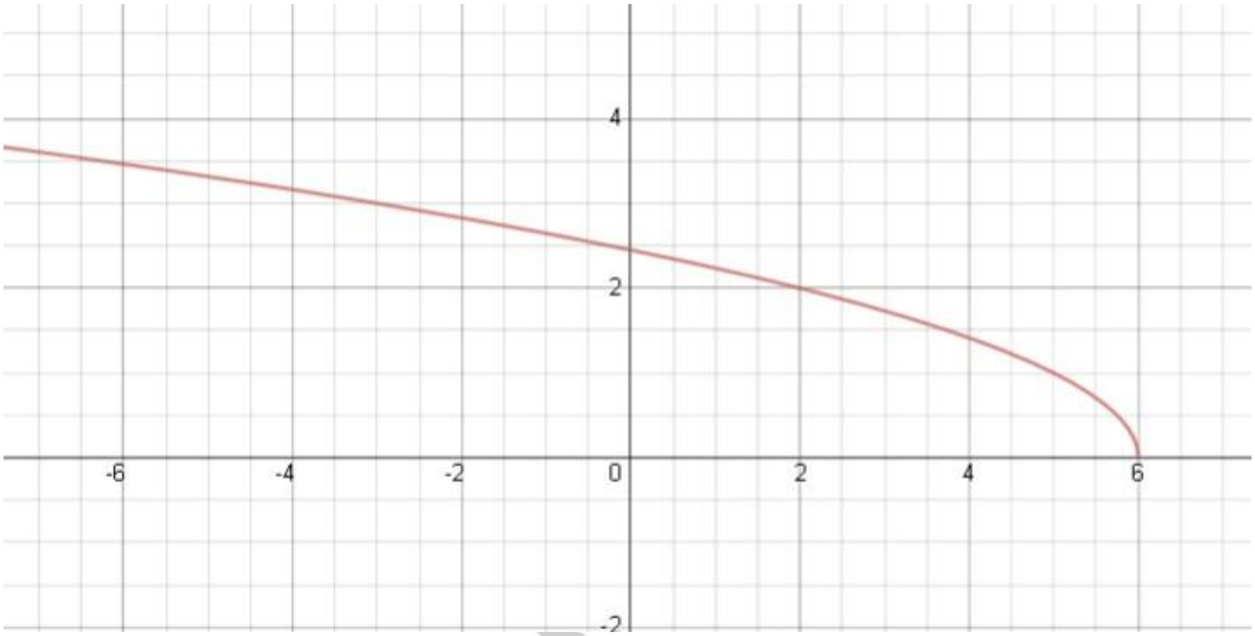
We can factor out $36 - x^2$: $(6 + x)(6 - x)$

Let's divide it up and see $(6 + x)$ within the domain of f (which is $\sqrt{x}$):



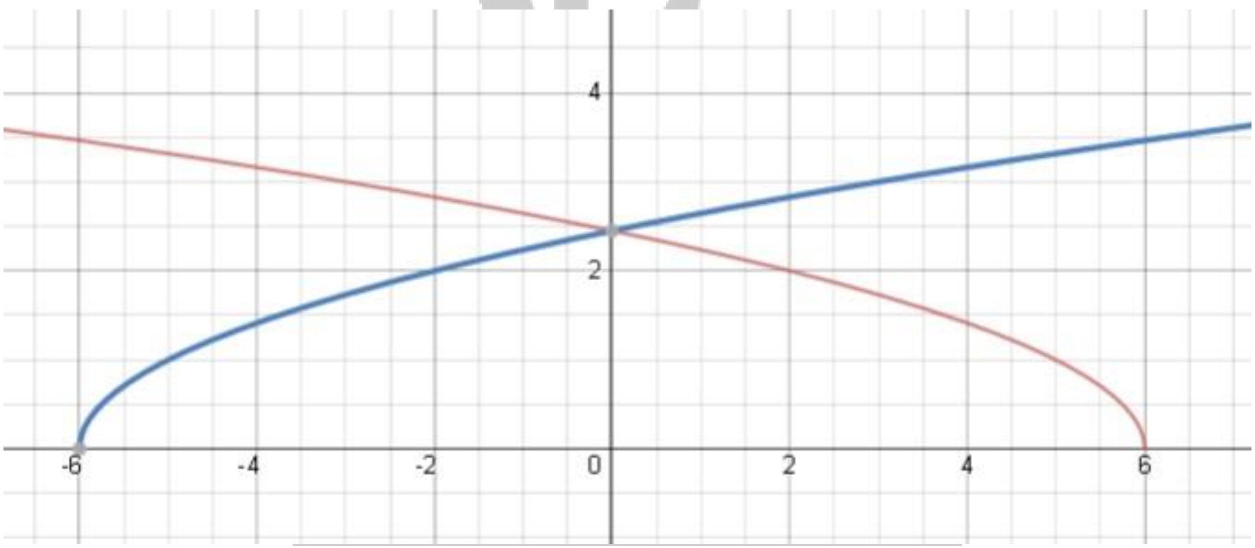
The domain is $[-6, \infty)$.

Now we look at $(6 - x)$ within the domain of f:

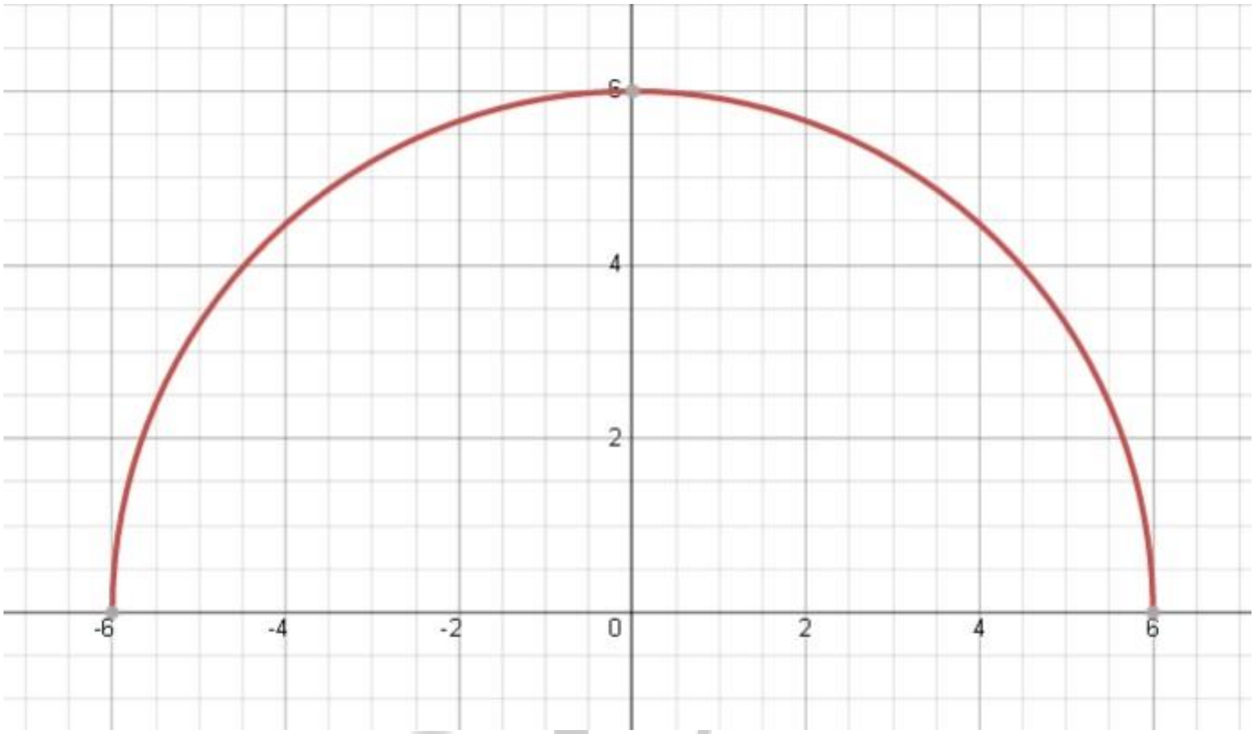


The domain is (-∞, 6].

When we graph them together, we see the overlap:



When we graph our composite function:



The domain of the composite function is [-6, 6]. From the properties of continuity, $g(x)$ is a polynomial and is continuous at every point in its domain. For $f(x)$, the values inside the radical had to be zero or positive. **The domain of *g* that is within *f* had to be zero or positive**. When we plug in any x that is within the interval of [-6, 6] we would get zero or a positive value.

The function is continuous on this closed interval. You can review the concepts of continuity on a closed interval from the previous section and you can see that our composite function is continuous in the open interval (-6, 6) and continuous from the right and from the left.

Intermediate Value Theorem

The **Intermediate Value Theorem** states that if a function is continuous on the closed interval [a, b], and $f(a)$ does not equal $f(b)$, then there is at least one number c that exists in the closed interval [a, b] such that $f(c) = k$. "k" would be any real number located between $f(a)$ and $f(b)$.

A good example would be height in regards to human growth and development. If a child is 4 feet 11 inches (≈ 149.86 cm) at 11 years of age and 5 feet 10 inches (≈177.8 cm) when the person is 20 years of age, we can say that for any height between 4'11" and 5'10" during those 9 years, there was a time when at time ***t*** the height ***h*** would be a value exactly on that interval. In other words, between 11 years old and 20 years old, the individual had a height between 4'11" (150 cm) and 5'10" (178 cm). Since growth is continuous and values don't change arbitrarily as one ages, this is an applicable example.

Let's take a look at an example:

Apply the Intermediate Value Theorem to show that there exists a number *c* such that *f*(*c*) = 40 in the interval [0, 5] if the polynomial *f*(*x*) = $x^3 + 4x - 1$.

Let's assume that the polynomial is continuous.

When we calculate the values in the interval:

$$\boldsymbol{f(0) = 0^3 + 4(0) - 1}$$
$$\mathbf{= -1}$$

$$\boldsymbol{f(5) = 5^3 + 4(5) - 1}$$

$$\mathbf{= 125 + 20 - 1}$$

$$\mathbf{= 144}$$

Since the number "40" falls between -1 and 144, we can deduce that there is a **c** where *f*(**c**) = 40.

Infinite Limits

In the earlier section where we discussed limits that failed to exist, we looked at #2 which discussed limits where *f*(*x*) increases or decreases without bound as *x* approaches *c*. These limits are referred to as **infinite limits**.

When determining infinite limits, first check that **the value of the numerator <u>does not</u> become 0 and the denominator <u>does</u> become 0** when you plug in **c** for the ***x***-values. For example:

$$\lim_{x \to 4^+} \frac{x}{x-4}$$

When we evaluate the limit we get:

$$= \frac{4}{4-4}$$

$$= \frac{4}{0}$$

Next, we will check whether *f*(*x*) increases without bound as *x* approaches *c* (to ∞) or decreases without bound as *x* approaches *c* (to -∞).

The first thing we need to check is whether *x* approaches *c* from the left or from the right. If *x* approaches *c* from the left it will be noted as ($\boldsymbol{x \to c^-}$) and if *x* approaches c from the right it will be noted as ($\boldsymbol{x \to c^+}$).

Since our example notes ($\boldsymbol{x \to 4^+}$), we will be choosing values close to 4 from the right. If you recall from our table of values, we will use values approaching **c** from the right (**c** + .01; **c** + .001; **c** + .0001). In our case, we can choose values close to 4 from the right such as 4.01, 4.001, or 4.0001.

*If our example approached **c** from the left ($\boldsymbol{x \to c^-}$), we will instead use (values such as **c** - .01, **c** - .001, or **c** - .0001).*

Now back to our example. After you choose your value, plug it in to the numerator and denominator. You don't need to worry about the precise value. **All you need to focus on is whether it is positive or negative**.

For our example, let's choose 4.001:

Numerator: 4.001 (**positive**)

Denominator: 4.001 - 4 = .001 (**positive**)

$$\frac{4.001}{4.001 - 4}$$

$$= \frac{4.001}{.001}$$

$$= \frac{+}{+}$$

Since performing the above operation will lead to a <u>positive value</u>, you can conclude:

$$\lim_{x \to 4^+} \frac{x}{x - 4} = \infty$$

Let's take a look at the infinite limit when *x* approaches *c* from the left:

$$\lim_{x \to 4^-} \frac{x}{x-4}$$

We follow the same steps above, but instead we will select values that are close to 4 from the left. Since we used **c** + .001 when we examined the limit from the right, we will use **c** - .001 when we examine the limit from the left.

4 - .001 = 3.999

Let's now plug in this value (where *x* = *c*):

Numerator: 3.999 (**positive**)

Denominator: 3.999 - 4 = -.001 (**negative**)

$$\frac{3.999}{3.999-4}$$

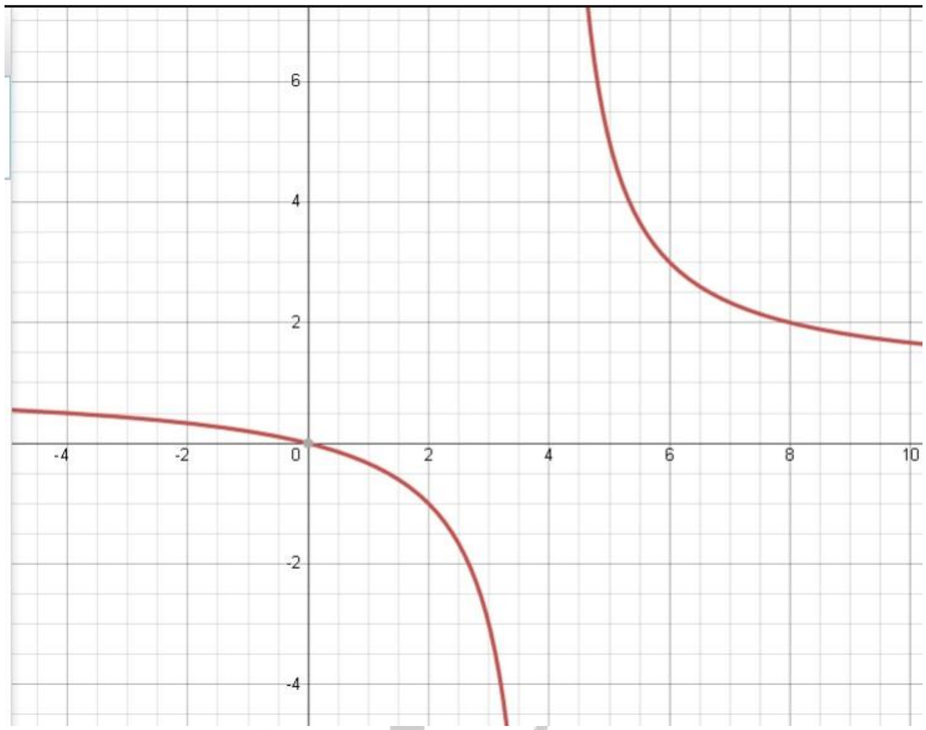
$$= \frac{3.999}{-.001}$$

$$= \frac{+}{-}$$

Since performing the above operation will lead to a <u>negative value</u>, you can conclude:

$$\lim_{x \to 4^-} \frac{x}{x-4} = -\infty$$

When we graph our function, we can see the behavior of *f*(*x*) as *x* approaches 4:



This shows us that the limit fails to exist since $f(x)$ does not approach a finite number.

Vertical Asymptotes

When we looked at the graph in the previous section, we saw that as x approached c from the left and from the right, the graph of $f(x)$ moved closer and closer to $\boldsymbol{x = 4}$.

The line where x = c is referred to as the vertical asymptote of the graph of $f(x)$. As $f(x)$ approaches infinity or negative infinity as x approaches c from either side (left or right), the line where $x = c$ is the vertical asymptote of the graph of the function.

To determine the vertical asymptote we apply what we learned from the previous section:

For the graph of the function:

$$\boldsymbol{h(x) = \frac{f(x)}{g(x)}}$$

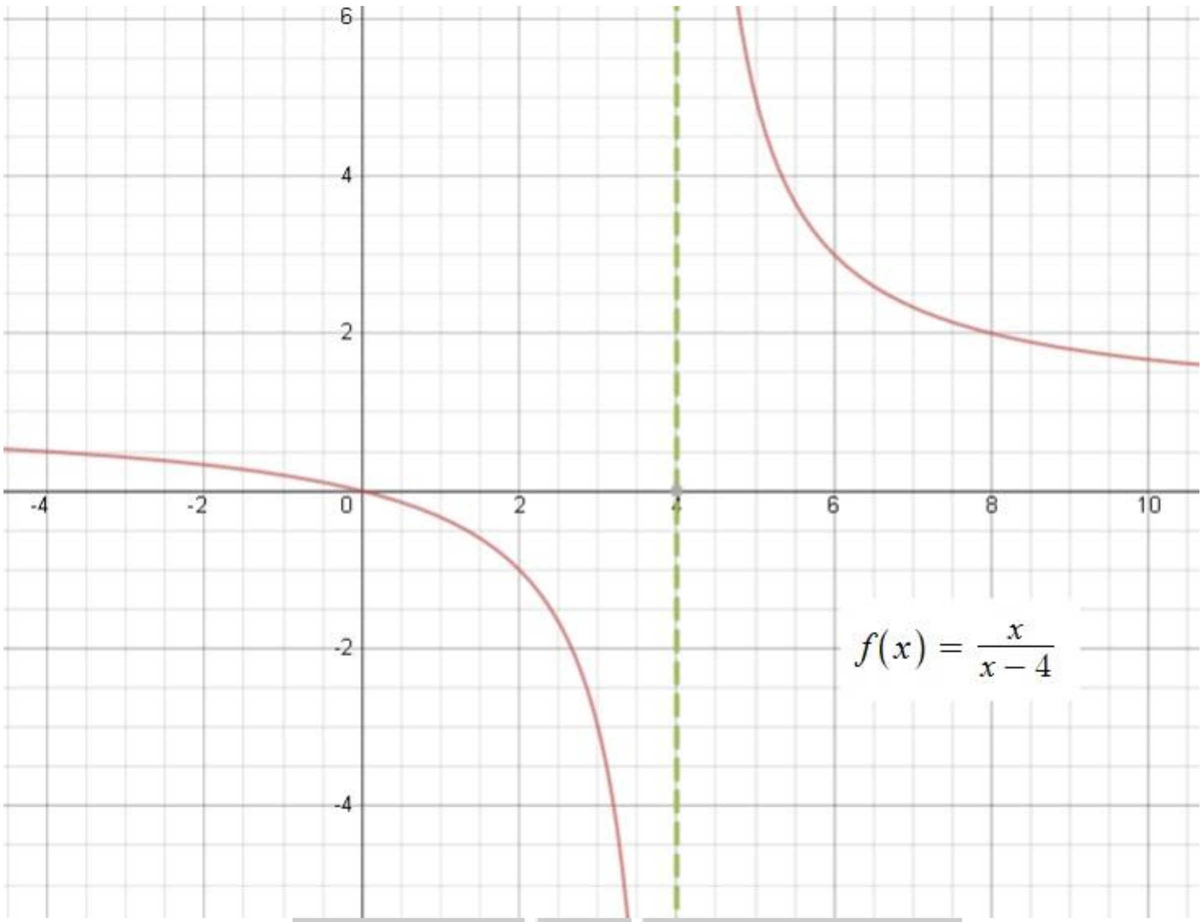
where $f(x)$ and $g(x)$ are continuous on an open interval containing c

if the numerator $f(c)$ does not equal 0 (i. e. $f(c) \neq 0$), the denominator $g(c)$ does equal 0 (i. e. $g(c) = 0$), and there is an open interval which contains c such that $g(x) \neq 0$ for all $x \neq c$ in the interval.

Let’s do an example to find the vertical asymptote.

In our previous example, we saw that the function had one vertical asymptote:

$$f(x) = \frac{x}{x-4}$$



The vertical asymptote is at $x = 4$.

Remember that if a function f has a vertical asymptote at $x = c$, the function is not continuous at c.

Example: Find the vertical asymptote of the graph of the function:

$$g(x) = \frac{x+6}{x^2+3x-4}$$

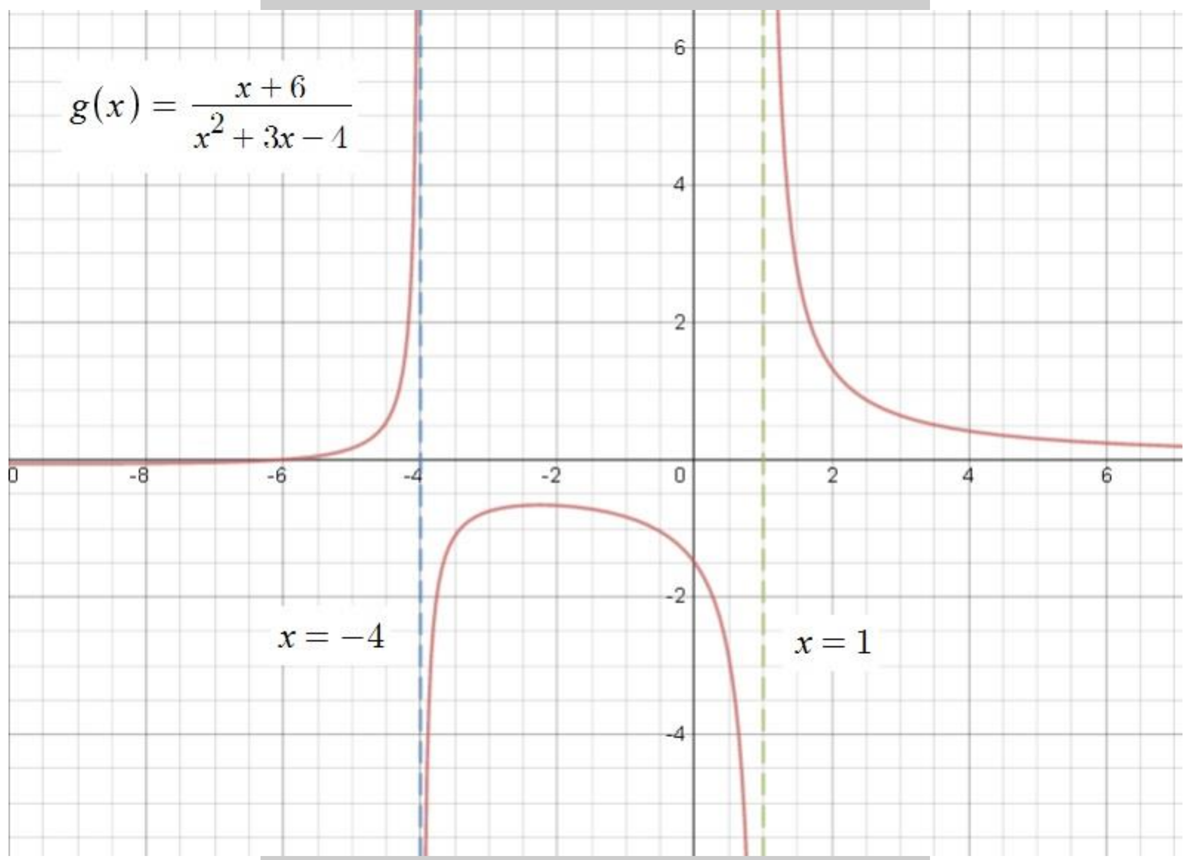
The first thing we need to do is simplify the expression. We can factor out the denominator:

$$g(x) = \frac{x+6}{x^2+3x-4}$$

$$= \frac{x+6}{(x+4)(x-1)}$$

After factoring out the denominator, we can determine that the denominator will be 0 at x = -4 and x = 1. When we plug in x = -4 and x = 1 in the numerator, it does not equal 0.

Our conclusion is that the graph of the function *g* has two vertical asymptotes: *x* = -4 and *x* = 1.



Properties of Infinite Limits

Let f and g be functions and let c and L be real numbers:

$$\lim_{x \to c} f(x) = \infty$$
and
$$\lim_{x \to c} g(x) = L$$

Sum or Difference:

$$\lim_{x \to c}[f(x) \pm g(x)] = \infty$$

Product:

$$\lim_{x \to c}[f(x)g(x)] = \infty, L > 0$$

$$\lim_{x \to c}[f(x)g(x)] = -\infty, L < 0$$

Quotient:

$$\lim_{x \to c}\frac{g(x)}{f(x)} = 0$$

Similar properties also apply for one-sided limits and when the limit of the function as x approaches c is negative infinity ($\lim_{x \to c} f(x) = -\infty$).

Limit Definition of the Derivative

As we transition into the next chapter, we will learn important differentiation rules. We will finish this handout by looking at the limit definition of the derivative.

$$\lim_{\Delta x \to 0}\frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Example:
Find the derivative of the function using the limit definition.

$$f(x) = 4x^2 - 9x$$

In our example for $f(x + \Delta x)$, we will put $x + \Delta x$ where x is to get:

$$f(x+\Delta x) = 4(x+\Delta x)^2 - 9(x+\Delta x)$$

$$\lim_{\Delta x \to 0}\frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{[4(x+\Delta x)^2 - 9(x+\Delta x)] - [4x^2 - 9x]}{\Delta x}$$

Note:
$(\boldsymbol{x} + \Delta \boldsymbol{x})^2 = \boldsymbol{x}^2 + 2\boldsymbol{x}\Delta \boldsymbol{x} + (\Delta \boldsymbol{x})^2$

$$= \lim_{\Delta x \to 0} \frac{4[x^2 + 2x\Delta x + (\Delta x)^2] - [9x + 9\Delta x] - [4x^2 - 9x]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[4x^2 + 8x\Delta x + 4(\Delta x)^2] - 9x - 9\Delta x - 4x^2 + 9x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{8x\Delta x + 4(\Delta x)^2 - 9\Delta x}{\Delta x}$$

Factor out the Δx and divide out:

$$= \lim_{\Delta x \to 0} \frac{\Delta x(8x + 4\Delta x - 9)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 8x + 4\Delta x - 9$$

Evaluate the limit:

$= 8x + 4(0) - 9$

$= \mathbf{8}\boldsymbol{x}\ \mathbf{- 9}$

When we learn the differentiation rules in the next chapter such as the Constant Multiple Rule and the Power Rule, we can simply apply them to cases such as our above example:

$$f(x) = 4x^2 - 9x$$

$$f'(x) = 4\frac{d}{dx}[x^2] - 9\frac{d}{dx}[x]$$

$$= 4(2x) - 9(1)$$

$= \mathbf{8}\boldsymbol{x}\ \mathbf{- 9}$

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